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Normal Variance-Mean Mixtures and *z* Distributions

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Summary

A survey is given of general properties of normal variance-mean mixtures, including various new results. In particular, it is shown that the class of self-reciprocal normal variance mixtures is rather wide, and some Tauberian results are established from which relations between the tail behaviour of a normal variance-mean mixture and its mixing distribution may be deduced. The generalized hyperbolic distributions and the modulated normal distributions provide examples of normal variance-mean mixtures whose densities can be given in terms of well-known functions, and it is proved that also the *z* distributions, i.e. the class of distributions generated from the beta distribution through logistic transformation followed by introduction of location and scale parameters, are normal variance-mean mixtures. (The *z* distributions include the hyperbolic cosine distribution and the logistic distribution.) Some properties of the associated mixing distributions are derived, and the *z* distributions are shown to be self-decomposable.

Key words: First hitting times; Hyperbolic cosine distribution; Hyperbolic distributions; Logistic distribution; Modulated normal distributions; Self-decomposability; Self-reciprocity; Tail behaviour; Tauberian propositions; Thorin class.

1 Introduction

Mixtures of normal distributions play an increasingly important role in both the theory and the practice of statistics. In particular, such mixtures occur as typical limit distributions in asymptotic theory for dependent random variables and they are useful for analysing data from a variety of heavy-tailed and skew empirical distributions. In the present paper we are concerned with normal variance-mean mixtures, and in particular normal variance mixtures, with a continuous type mixing distribution.

We say that the distribution of an r-dimensional random vector x is a *normal variance-mean mixture* with mixing distribution F provided that x, for a given $u \geqslant 0$, follows an r-dimensional normal distribution with covariance matrix $u\Delta$ and mean vector $\mu + u\beta$, and provided u follows a probability distribution F on $[0, \infty)$. In symbols: $x \mid u \sim N_r(\mu + u\beta, u\Delta)$ and $u \sim F$. Here Δ denotes a constant, positive-definite $r \times r$ matrix and μ and β are constant vectors of dimension r. If $\beta = 0$ we have a *normal variance mixture*.

We shall discuss certain general properties of normal variance-mean mixtures and also some families of distributions of this type which are of special interest. Section 2 gives a review of known results on normal variance-mean mixtures and contains some new observations as well. Among the general properties discussed are infinite divisibility, in particular self-decomposability, and self-reciprocity, while the family of generalized hyperbolic distributions provides some main exemplifications. In § 3 the family of z

distributions, i.e. the distributions on the real line having probability density functions of the form

$$g(x)=\frac{1}{\sigma B(\alpha,\beta)}\frac{\{\exp[(x-\mu)/\sigma]\}^{\alpha}}{\{1+\exp[(x-\mu)/\sigma]\}^{\alpha+\beta}},$$

where B denotes the beta distribution, are shown to be self-decomposable normal variance-mean mixtures and their mixing distributions are determined. Some properties of these mixing distributions are discussed in § 4. The final § 6 contains a general discussion of the relation between the tail-behaviour of a normal variance-mean mixture and the tail-behaviour of its mixing distribution. This discussion is based on some Tauberian results established in § 5.

2 Normal variance-mean mixtures

In the present section we survey the general properties of normal variance-mean mixtures and some well-studied examples of such distributions, namely the generalized hyperbolic distributions and the modulated normal distributions. As a special point we delineate a broad class of self-reciprocal distributions.

Definition 2.1. Suppose x is a random vector which, for a given $u \geq 0$, follows an r-dimensional normal distribution with covariance matrix $u\Delta$ and mean vector $\mu+u\beta$, where Δ is a symmetric, positive-definite $r\times r$-matrix with determinant one, and μ and β are vectors of dimension r. Suppose moreover that u follows a probability distribution F on $[0,\infty)$. Then we say that the distribution of x is a *normal variance-mean mixture* with position μ, drift β, structure matrix Δ and mixing distribution F. If $\beta=0$ we speak of a *normal variance mixture.*

The condition $|\Delta|=1$ is imposed only to avoid an unidentifiable scale factor. Our vectors are row vectors and transposition is indicated by $'$ as upper index.

As a possible model behind a normal variance-mean mixture with $\Delta=I$ one has the position of an r-dimensional Brownian motion with drift β, started at μ, and observed at a random time u. A related application is given by Barndorff-Nielsen & Darroch (1981). A normal variance mixture may be thought of in the following way. Suppose the random variable $y\sim N_r(0,\Sigma)$ is independent of $u\geq 0$, and that $u\sim F$. Then the distribution of $u^{\frac{1}{2}}y$ is a normal variance mixture.

In many cases, if an explicit representation of a distribution as a normal variance-mean mixture can be found then this can be taken to advantage in simulating the distribution; see, for instance, Atkinson (1979), Andrews & Mallows (1974) and Relles (1970).

We will now state some elementary results about normal variance-mean mixtures. Suppose the distribution P of x is a normal variance-mean mixture with position μ, drift β, structure matrix Δ, and mixing distribution F, and that $u\sim F$.

The random vector x has probability density function

$$g(x)=\exp\{(x-\mu)\Delta^{-1}\beta'\}\int(2\pi u)^{-r/2}\exp\{-\tfrac{1}{2}(x-\mu)(u\Delta)^{-1}(x-\mu)'-\tfrac{1}{2}u\beta\Delta^{-1}\beta'\}F(du), \tag{2.1}$$

and characteristic function

$$\hat{g}(\theta)=e^{i\theta\mu'}\varphi(i\theta\beta'-\tfrac{1}{2}\theta\Delta\theta'), \tag{2.2}$$

where φ is the moment generating function of F, that is $\varphi(s)=E\{\exp(su)\}$. The distribution of x is isotropic if and only if $\beta=0$ and $\Delta=I$.

Suppose that the first moment of F exists, then $Ex = \mu + \beta Eu$. If u has variance $Vu < \infty$ then x has covariance matrix $Eu\Delta + Vu\beta'\beta$.

Let (x_1, x_2) be a partition of x such that x_1 is k-dimensional ($k < r$), and let (μ_1, μ_2), (β_1, β_2) and

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$$

be the corresponding partitions of μ, β and Δ. Then the distribution of x_1 is a normal variance-mean mixture with position μ_1, drift $|\Delta_{11}|^{-1/k}\beta_1$, structure matrix $|\Delta_{11}|^{-1/k}\Delta_{11}$, and with the distribution of $|\Delta_{11}|^{1/k}u$ as mixing distribution; and the conditional distribution of x_1 given x_2 is a normal variance-mean mixture with position $\mu_1 + (x_2 - \mu_2)\Delta_{22}^{-1}\Delta_{21}$, drift $\nu^{-1/k}(\beta_1 - \beta_2\Delta_{22}^{-1}\Delta_{21})$, structure matrix $\nu^{-1/k}(\Delta_{11} - \Delta_{12}\Delta_{22}^{-1}\Delta_{21})$, and the conditional distribution of $\nu^{1/k}u$ given x_2 as mixing distribution. Here $\nu = |\Delta_{11} - \Delta_{12}\Delta_{22}^{-1}\Delta_{21}|$. Furthermore, if A is a regular matrix and b is a vector then $xA' + b$ is a normal variance-mean mixture with position $\mu A' + b$, drift $|A|^{-2/k}\beta A'$, structure matrix $|A|^{-2/k}A\Delta A'$ and mixing distribution $|A|^{2/k}u$.

If x is a normal variance mixture with mixing distribution F and structure matrix I, then the canonical parameter domain of the linear exponential family $\mathcal{P}$ generated by x is $\{\theta : \frac{1}{2}\theta\theta' \in \Gamma\}$, where $\Gamma = \{\gamma \in \mathbb{R} : \varphi(\gamma) < \infty\}$ is the canonical parameter domain of the linear exponential family $\mathcal{F}$ generated by F. Furthermore, any $P_\theta \in \mathcal{P}$ is a normal variance-mean mixture with drift θ, structure matrix I, and mixing distribution $F_{\frac{1}{2}\theta\cdot\theta} \in \mathcal{F}$. Here P_θ is given by the density

$$g_\theta(x) = e^{\theta x'}g(x)/\varphi(\tfrac{1}{2}\theta \cdot \theta), \quad F_\gamma(du) = e^{\gamma u}F(du)/\varphi(\gamma).$$

Infinite divisibility of F implies that P is infinitely divisible, as follows immediately from (2.2). See also Kent (1981) and Kelker (1971).

Self-decomposability of F is not, in general, sufficient to ensure self-decomposability of P, but it can be shown that if F is self-decomposable then P is self-decomposable provided $\beta = 0$. Moreover, one-dimensional variance-mean mixtures are self-decomposable if the mixing distribution F has the stronger property of belonging to the Thorin class of 'generalized gamma convolutions'; it can even be deduced that P belongs to the extended Thorin class. See Thorin (1978) and Halgreen (1979).

The property of self-decomposability is of some special statistical interest because only self-decomposable distributions can occur as the one-dimensional marginal distributions of stationary autoregressive schemes, see Bondesson (1981b).

A distribution is called self-reciprocal if its density function and its characteristic function are proportional. Examples are: the normal distribution, the hyperbolic cosine distribution (Feller, 1971, pp. 502–503), and the r-dimensional generalized hyperbolic distribution with $\lambda = r/4$, $\beta = \mu = 0$ and $\kappa = \delta$. We shall return to the two latter distributions later. All three of these distributions are normal variance mixtures, so it seems to be of some interest to establish a necessary and sufficient condition for a normal variance mixture to be self-reciprocal. Suppose $\beta = \mu = 0$, $\Delta = I$, and $E(u^{-r/2}) < \infty$. Then we can define a probability measure $\tilde{F}$ on $(0, \infty)$ by $d\tilde{F} = cu^{-r/2}\,dF$ where $c^{-1} = E(u^{-r/2})$. If $v \sim \tilde{F}$ then it can easily be seen, by comparing (2.1) and (2.2), that $\hat{g}(x) = c_0 g(x)$ for some constant c_0 if and only if $v^{-1} \sim F$. Hence, if F has density f with respect to Lebesgue measure on $(0, \infty)$ then P is self-reciprocal if and only if

$$f(u) = cu^{\frac{1}{2}(r-4)}f(u^{-1}).$$

From this relation and from the definition of c it follows that $c = 1$. Hence P is self-reciprocal if and only if

$$f(u)=u^{\frac{1}{2}(r-4)}f(u^{-1}). \tag{2.3}$$

It follows that the class of self-reciprocal normal variance mixtures with $\Delta=I$ and absolutely continuous mixing distribution is generated in the following way. Any positive, integrable function on the interval $[1,\infty)$ can be extended to a function on $(0,\infty)$ using (2.3) to define it on (0, 1). This extended function is integrable and can be normalized to integrate to 1, that is to become a probability density function. The normal variance mixture with structure matrix I and this density function as mixing distribution is self-reciprocal. Hence, the class of self-reciprocal normal variance mixtures is rather wide.

A fairly broad class of normal variance-mean mixtures is the class of r-dimensional *generalized hyperbolic distributions* (Barndorff-Nielsen, 1977, 1978); see also Barndorff-Nielsen & Blaesild (1980), and Blaesild & Jensen (1980). These distributions are obtained as the variance-mean mixtures of r-dimensional normal distributions with position μ, drift $\beta\Delta$, and structure matrix Δ, the mixing distributions being the generalized inverse Gaussian distributions. Specifically, if the mixing distribution is the generalized inverse Gaussian distribution $N^{\dagger}(\lambda, \delta^2, \kappa^2)$ with probability density function

$$f(u)=\frac{(\kappa/\delta)^{\lambda}}{2K_{\lambda}(\delta\kappa)}u^{\lambda-1}\exp\{-\tfrac{1}{2}(\delta^2u^{-1}+\kappa^2u)\}, \tag{2.4}$$

where $u>0$, K_{λ} is a Bessel function, and

$$\begin{aligned}\delta\geq 0, \kappa>0 \quad &\text{for} \quad \lambda>0,\\ \delta>0, \kappa>0 \quad &\text{for} \quad \lambda=0,\\ \delta>0, \kappa\geq 0 \quad &\text{for} \quad \lambda<0,\end{aligned}$$

then the resulting normal variance-mean mixture has probability density

$$\frac{(\kappa/\delta)^{\lambda}}{(2\pi)^{r/2}K_{\lambda}(\delta\kappa)}\frac{K_{\lambda-r/2}(\alpha\{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'\}^{\frac{1}{2}})}{(\{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'\}^{\frac{1}{2}}/\alpha)^{r/2-\lambda}}\exp\{\beta(x-\mu)'\}, \tag{2.5}$$

where $\alpha^2=\kappa^2+\beta\Delta\beta'$.

For $\lambda=\frac{1}{2}(r+1)$ and $\lambda=\frac{1}{2}(r-1)$ the expression (2.5) takes a simple explicit form. For $\lambda=\frac{1}{2}(r+1)$ the r-dimensional hyperbolic distribution is obtained and (2.5) becomes

$$\frac{(\kappa/\delta)^{\frac{1}{2}(r+1)}}{(2\pi)^{\frac{1}{2}(r-1)}2\alpha K_{\frac{1}{2}(r+1)}(\delta\kappa)}\exp[-\alpha\{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'\}^{\frac{1}{2}}+\beta(x-\mu)']. \tag{2.6}$$

The graph of the log density function is a hyperboloid. For various applications of the one- and two-dimensional hyperbolic distributions, see Bagnold & Barndorff-Nielsen (1980), Barndorff-Nielsen (1977, 1979), Barndorff-Nielsen, Dalsgaard, Halgreen, *et al.* (1982) and Blaesild (1981). The 3-dimensional version of this distribution occurs in relativistic statistical mechanics (Chandrasekhar, 1957; Barndorff-Nielsen, 1982; de Groot, van Leeuwen & van Weert, 1980). For $\lambda=\frac{1}{2}(r-1)$ we get

$$\frac{(\kappa/\delta)^{\frac{1}{2}(r-1)}}{(2\pi)^{\frac{1}{2}(r-1)}2K_{\frac{1}{2}(r-1)}(\delta\kappa)}\frac{\exp[-\alpha\{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'\}^{\frac{1}{2}}+\beta(x-\mu)']}{\{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)'\}^{\frac{1}{2}}}. \tag{2.7}$$

This can be regarded as a distribution on a hyperboloid in $\mathbb{R}^{r+1}$. It possesses a number of useful mathematical and statistical properties, (Barndorff-Nielsen, 1978; Jensen, 1981).

If $\lambda<0$ and $\kappa=0$ then (2.4) is the distribution of the reciprocal of a gamma distributed random variable and if, moreover, $\mu=\beta=0$ and $\Delta=I$ then (2.5) is the r-dimensional t distribution. For $\lambda>0$ and $\delta=0$, expression (2.4) is a gamma distribution and, for $r=1$, the distribution (2.5) is McKay's Bessel function distribution (Johnson & Kotz, 1970). If, in particular, $\lambda = 1$ we have the skew Laplace distribution, and for $\beta = 0$ the usual Laplace distribution. Moreover, the normal distribution lies on the boundary of the family (2.5), because (2.4) tends to a distribution concentrated at $c \in (0, \infty)$ if $\delta, \kappa \to \infty$ such that $\delta/\kappa \to c$. Finally, a generalized inverse Gaussian distribution is obtained as the limit when $\lambda > 0$, $\alpha - \beta = \gamma$, and $\delta \to 0$ and $\kappa \to \infty$ such that $\kappa^2\delta^2 \to c$, where c and γ are constants. The limit distribution is $N^\dagger(\lambda, c/2, 2\gamma)$.

The characteristic function of the generalized hyperbolic distribution is

$$e^{i\theta\mu'}\left\{\frac{\kappa^2}{\kappa^2 - 2i\theta\Delta\beta' + \theta\Delta\theta'}\right\}^{\lambda/2} \frac{K_\lambda(\delta(\kappa^2 - 2i\theta\Delta\beta' + \theta\Delta\theta')^{\frac{1}{2}})}{K_\lambda(\delta\kappa)}.$$

As previously noted, if $\lambda = r/4$, $\mu = \beta = 0$ and $\kappa = \delta$ then the distribution is self-reciprocal.

From the general results given above it follows that the family of generalized hyperbolic distributions is closed under margining, conditioning with marginals, and affine transformations.

It was established by Bondesson (1979) and Halgreen (1979) that (2.4) belongs to the Thorin class. From the general results on variance-mean mixtures mentioned above it follows that for $r = 1$ the distributions (2.5) are self-decomposable, and that they are self-decomposable in any dimension if $\beta = 0$. For $r > 1$ and $\beta \neq 0$ it was proved by Shanbhag & Sreehari (1979) that the generalized hyperbolic distributions are not in general self-decomposable according to the restricted, homothetical definition of self-decomposability of multivariate distributions. All of the generalized hyperbolic distributions are, however, infinitely divisible.

In § 3 we shall discuss another specific class of normal variance-mean mixtures, namely the z distributions, in some detail.

Romanowski (1979) gives an intuitive argument why normal variance mixtures are capable of describing the variation in so many real data sets. The argument connects the mixing process to a random number of active elementary errors. The argument can be made formally correct by the result that under weak conditions the sum of N_n random variables, where N_n is a random natural number, is asymptotically distributed as a normal variance mixture as $n \to \infty$ provided N_n/n converges in probability to a random variable u; see Rényi (1960). If the elementary errors are allowed a nonzero mean the argument generalizes to an argument for the use of normal variance-mean mixtures.

Romanovski put forward this argument in connection with a study of what he has termed *modulated normal distributions* which are in fact particular instances of normal variance mixtures. Suppose that $x \mid u \sim N(0, \frac{1}{2}\kappa^2 u)$. Then the type I modulated normal distributions are obtained for $u \sim B(a, 1)$, where B denotes the beta distribution and $a > 0$. The probability density function of x is

$$\frac{a}{\kappa\sqrt{\pi}}\left(\frac{x}{\kappa}\right)^{2a-1}\Gamma\left(-(a-\tfrac{1}{2}), \left(\frac{x}{\kappa}\right)^2\right),$$

where Γ denotes the complementary incomplete gamma function. If, instead, u follows a Pareto distribution on $[1, \infty)$ with parameter a, then the variance mixture has density

$$\frac{a}{\kappa\sqrt{\pi}}\left(\frac{x}{\kappa}\right)^{-(2a+1)}\gamma\left(a+\tfrac{1}{2}, \left(\frac{x}{\kappa}\right)^2\right);$$

here γ is the incomplete gamma function. This is the type II modulated normal distribution. Since the Pareto distribution is self-decomposable (Thorin, 1977), it follows that the type II modulated normal distribution is self-decomposable.

Recently, there has been some considerable interest in multivariate distributions with spherical or elliptical contours; see Bishop, Fraser & Ng (1979), Muirhead (1980), Cambanis, Huang & Simons (1981), Chmielewski (1980, 1981), Eaton (1981), Jensen (1981), Jensen & Good (1981), Letac (1981) and Smith (1981). Most useful distributions of this kind are normal mixtures.

Finally, it is pertinent to mention that mixtures of normal distributions occur as limiting distributions in generalizations of the central limit problem to nonindependent summands, see for instance Hall (1977), Rootzén (1977) and Shiryayev (1981). It is a closely related fact that the (unconditional) limiting distribution of the maximum likelihood estimator in so-called nonergodic stochastic processes is generally a normal mixture; see Hall & Heyde (1980) and Feigin (1981).

3 The z distributions

The z distribution with parameters α, β, σ and μ will be denoted $z(\alpha, \beta, \sigma, \mu)$. Its density with respect to the Lebesgue measure is given by

$$g(x)=\frac{1}{\sigma B(\alpha,\beta)}\frac{\{\exp[(x-\mu)/\sigma]\}^{\alpha}}{\{1+\exp[(x-\mu)/\sigma]\}^{\alpha+\beta}} \quad (x\in\mathbb{R};\ \alpha,\beta,\sigma>0;\ \mu\in\mathbb{R}), \tag{3.1}$$

where B is the beta function. We propose to use the term z distributions because of the origin the distributions have in Fisher's work relating to the so-called z transformations; see below. In the following we survey the z distributions and present some new properties of these. In particular, we show that the z distributions are normal variance-mean mixtures and we specify their mixing distributions.

The z distribution has log linear tails. More specifically, if the density function is plotted with a logarithmic scale for the ordinate axis then the lower tail tends asymptotically to a straight line with slope α/σ, while the slope of the asymptote of the upper tail is $-\beta/\sigma$. If $\alpha=\beta$ the distribution is symmetric, whereas it is negatively (positively) skewed if $\alpha>\beta$ ($\beta>\alpha$). The density function of the symmetric distribution $z(\delta, \delta, 1, 0)$, $\delta>0$, may be rewritten as

$$\{4^{\delta}B(\delta,\delta)\}^{-1}\{\cosh(x/2)\}^{-2\delta}, \tag{3.2}$$

and the linear exponential family generated by $z(\delta, \delta, 1, 0)$ is $z(\delta+\theta, \delta-\theta, 1, 0)$ for $|\theta|<\delta$. Note that if μ and σ are constant then the class of z distributions is a regular exponential family of order 2.

The characteristic function of the z distribution is

$$\hat{g}(t)=\frac{e^{it\mu}B(\alpha+i\sigma t, \beta-i\sigma t)}{B(\alpha,\beta)}. \tag{3.3}$$

In particular, the hyperbolic cosine distribution, which equals $z(\frac{1}{2}, \frac{1}{2}, 1/\sqrt{2\pi}, 0)$ and whose density is

$$\frac{1}{\sqrt{2\pi}\cosh(\sqrt{\pi/2}\,x)}$$

has characteristic function

$$\frac{1}{\cosh(\sqrt{\pi/2}\,t)}.$$

If X is beta distributed with parameters α and β then $\log(X/(1-X))\sim Z(\alpha,\beta,1,0)$. Hence, if X is F distributed with f_1 and f_2 degrees of freedom then $\log X\sim z(\frac{1}{2}f_1, \frac{1}{2}f_2, 1, \log(f_2/f_1))$; see Fisher (1935).

The z distributions appeared for the first time in Fisher (1921), where it is shown that the distribution of the z transformation

$$z = \tfrac{1}{2} \log \{(1 + r(s-1))/(1-r)\}$$

of the intraclass correlation r derived from n sets of s normally distributed observations is $z(\frac{1}{2}(n-1), \frac{1}{2}n(s-1), \frac{1}{2}, \zeta + \varphi)$, where n is the sample size, $\tanh \varphi = (s-2)/s$ and ζ is the z transformation of the population correlation coefficient. Fisher also notes that if $n = 1$, $s = 2$ and the mean is assumed to be zero then $z \sim z(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \zeta)$. Furthermore it is shown that the z transformation of the interclass correlation is $z(\frac{1}{2}n - 1, \frac{1}{2}n - 1, \frac{1}{2}, 0)$ distributed, where $n (>2)$ is the sample size, provided the population correlation coefficient is zero.

The full location-scale class considered in the present paper was first treated by Prentice (1975), who used the fact that a number of common distributions or transformations of such distributions are included in the class (some of them as limits as parameters tend to infinity) to discriminate between these distributions. In particular, the log gamma model considered by Prentice (1974) is obtained as the weak limit of $z(\alpha, \beta, 1, \log(\beta/\alpha))$ for $\alpha \to \infty$ or $\beta \to \infty$. The logistic distribution ($\alpha = \beta = 1$), the generalized logistic distributions of Gumbel (1944) and Dubey (1969) (namely $z(m, m, 1, 0)$, $m \in \mathbb{N}$, and $z(1, \beta, \sigma, \mu)$) and the hyperbolic cosine distributions ($\alpha = \beta = \frac{1}{2}$) are all members of the class of z distributions. Also, the z distributions are implicitly mentioned by Johnson (1949). In order to compare his curve system to the Pearson system Johnson applies his S_B transformation to the beta distributions. According to the results given above the distributions obtained are in fact the z distributions. The moment generating function and the cumulants of the z distribution are given by Johnson & Kotz (1970, p. 152).

The z distributions seem to be useful as distributions of log income. Vartia & Vartia (1978) proposed to use the F distribution as the distribution of $(x - \tau)/\delta$, where $x \geqslant \tau$ is income and τ is known. This corresponds to using the $z(\alpha, \beta, 1, \mu)$ distribution to describe the distribution of $\log[(x - \tau)]$. In an earlier paper Vartia & Vartia (1972) proposed that the F distribution should be used as the distribution of $[(x-\tau)/\delta]^\gamma$, $\gamma > 0$, which corresponds to using the $z(\alpha, \beta, \sigma, \mu)$ distribution as log income distribution. Moreover, Singh & Maddala (1976) proposed the $z(1, \beta, \sigma, \mu)$ distribution as a distribution of log income. On the other hand the class of Champernowne distributions (Champernowne, 1952) has only two points of intersection with the z distributions, namely the logistic distribution and the hyperbolic cosine distribution.

We now prove that z distributions may be represented as normal variance-mean mixtures.

To make reference to the mixing distribution easier, we formulate the following definition.

Definition 3.1. Let $H(\delta, \gamma)$ denote the distribution on $(0, \infty)$ which has moment generating function

$$\varphi(s) = \prod_{k=0}^{\infty} \left\{1 - \frac{s}{\frac{1}{2}(\delta + k)^2 - \gamma}\right\}^{-1}, \tag{3.4}$$

where the parameters δ, γ satisfy $\delta > 0$, $\gamma < \frac{1}{2}\delta^2$.

Since $(1 - s/\lambda_0)^{-1}$ is the moment generating function of an exponential distribution with parameter $\lambda_0 > 0$, we see that the $H(\delta, \gamma)$ distribution lies in the class of *infinite convolutions of exponential distributions*, also known as the class of *Pólya distributions* with support $(0, \infty)$ (see, for example, Kent, 1980, p. 310).

We start by showing that the symmetric distribution $z(\delta, \delta, 1, 0)$ is a normal variance mixture, with mixing distribution $H(\delta, 0)$. Using (3.4) and (2.2), we see that the normal variance mixture based on $H(\delta, 0)$ has characteristic function

$$\hat{g}(t) = \prod_{k=0}^{\infty} (1 + t^2/(\delta + k)^2)^{-1}$$

which equals $\Gamma(\delta + it)\Gamma(\delta - it)/\Gamma(\delta)^2$ by formula 8.325-1 of Gradshteyn & Ryzhik (1965). Hence (see (3.3)) $\hat{g}(t)$ is precisely the characteristic function of $z(\delta, \delta, 1, 0)$, and the result follows.

To deal with the general z distributions, we first note that the linear exponential family generated by $H(\delta, 0)$ is $H(\delta, \gamma)$ where the canonical parameter domain for the parameter γ is $\Gamma = (-\infty, \frac{1}{2}\delta^2)$. Also, note that the linear exponential family generated by $z(\delta, \delta, 1, 0)$ is $z(\delta + \theta, \delta - \theta, 1, 0)$ for $|\theta| < \delta$. Hence, by a previous remark (in § 2), $z(\delta + \theta, \delta - \theta, 1, 0)$, $|\theta| < \delta$, is a normal variance-mean mixture with mixing distribution $H(\delta, \gamma)$, $\gamma = \frac{1}{2}\theta^2$. Performing a location-scale transformation we see that *any* z-distribution is a normal variance-mean mixture.

A series representation of the mixing densities can be given using the following result, which is extracted from Kent (1980).

LEMMA 3.1 (*Kent*, 1980). *If the moment generating function of an infinite convolution of exponential distributions is*

$$\varphi(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s}{\lambda_k}\right)^{-1},$$

where $0 < \lambda_1 < \lambda_2 < \dots$, *and if for all* $\varepsilon > 0$ *the condition*

$$|\text{Res}\,(\varphi, \lambda_k)| = 0(\exp(\varepsilon\lambda_k)) \quad (k \to \infty), \tag{3.5}$$

is satisfied, where

$$\text{Res}\,(\varphi, \lambda_k) = \left(\frac{d(1/\varphi)}{ds}\bigg|_{s=\lambda_k}\right)^{-1}$$

denotes the residue of $\varphi(s)$ *at* $s = \lambda_k$, *then the density function corresponding to* φ *is*

$$f(u) = -\sum_{k=1}^{\infty} \text{Res}\,(\varphi, \lambda_k) \exp(-\lambda_k u) \quad (u > 0).$$

For $H(\delta, 0)$ it follows by direct calculation using again formula 8.325-1 of Gradshteyn & Ryzhik (1965) that

$$\text{Res}\,(\varphi, \tfrac{1}{2}(\delta + k)^2) = -\binom{-2\delta}{k}(\delta + k)/B(\delta, \delta).$$

Since, for any $\varepsilon > 0$,

$$\left|\binom{-2\delta}{k}(\delta + k)\right| \leq (\delta + k)(2\delta + 1)^k = 0\left(\exp\left(\frac{\varepsilon}{2}(\delta + k)^2\right)\right),$$

we see that the condition (3.5) is satisfied and consequently the density of $H(\delta, 0)$ is

$$f(u) = \sum_{k=0}^{\infty} \binom{-2\delta}{k} \frac{(\delta + k)}{B(\delta, \delta)} \exp\{-\tfrac{1}{2}(\delta + k)^2 u\} \quad (u > 0). \tag{3.6}$$

Similarly the mixing distribution $H(\delta, \gamma)$, $\gamma = \frac{1}{2}\theta^2$, which leads to the asymmetric

$z(\delta+\theta, \delta-\theta, 1, 0)$ distribution, has the density

$$f(u)=\sum_{k=0}^{\infty}\binom{-2\delta}{k}\frac{(\delta+k)}{B(\delta+\theta, \delta-\theta)}\exp\{-\tfrac{1}{2}((\delta+k)^2-\theta^2)u\}. \tag{3.7}$$

It was noted above that $H(\delta, \gamma)$ is an infinite convolution of exponential distributions and hence $H(\delta, \gamma)$ belongs to the Thorin class for every (δ, γ). By results of Halgreen (1979) and Thorin (1978) this implies that the z distributions belong to the extended Thorin class and hence are self-decomposable. The fact that the z distributions belong to the extended Thorin class can also be deduced directly from a product representation of the characteristic function (3.3) given by formula 8.325-1 of Gradshteyn & Ryzhik (1965). *A fortiori*, the z distributions are infinitely divisible, a result which has previously been established for the hyperbolic cosine distribution and the logistic distribution; see Feller (1971) and Steutel (1979). Furthermore, as the extended Thorin class is closed under weak limits the log gamma model mentioned earlier belongs to the extended Thorin class.

It is also possible to prove directly that the log gamma distribution (and hence the z distributions) belong to the extended Thorin class; see Shanbhag & Sreehari (1977), Shanbhag, Pestana & Sreehari (1977), and Bondesson (1981a, p. 58). This result can easily be deduced from a product representation of the characteristic function of the log gamma distribution.

We may summarize these findings as follows.

THEOREM 3.1. *The z distributions are self-decomposable and are normal variance-mean mixtures, the mixing distributions being the $H(\delta, \gamma)$ distributions.*

The result that the logistic distribution is a normal variance mixture with the mixing distribution $H(1, 0)$ was given by Andrews & Mallows (1974).

4 Some properties of the mixing distribution

The mixing distribution $H(\delta, 0)$ is of interest in its own right, especially for $\delta=\frac{1}{2}$ and $\delta=1$.

When δ is an integer or a half-integer, the moment generating function of $H(\delta, 0)$ can be expressed by simple functions. If δ is an integer

$$\begin{aligned}\varphi(s)^{-1}&=\prod_{k=0}^{\infty}\left(1-\frac{s}{\frac{1}{2}(\delta+k)^2}\right)\\&=\prod_{k=1}^{\infty}\left(1-\frac{2s}{k^2}\right)\left\{(1-2s)\dots\left(1-\frac{2s}{(\delta-1)^2}\right)\right\}^{-1}\\&=\sin(\pi\sqrt{2s})\left\{\pi\sqrt{2s}(1-2s)\dots\left(1-\frac{2s}{(\delta-1)^2}\right)\right\}^{-1}\end{aligned}$$

(Gradshteyn & Ryzhik, 1965, formula 1.431-1), that is

$$\varphi(s)=\frac{\pi\sqrt{2s}(1-2s)\dots(1-2s/(\delta-1)^2)}{\sin(\pi\sqrt{2s})}.$$

In particular for $\delta=1$,

$$\varphi(s)=\pi\sqrt{2s}/\sin(\pi\sqrt{2s}).$$

Likewise if $\delta=(2m+1)/2$,

$$\varphi(s)=\begin{cases}1/\cos(\pi\sqrt{2s}) & (m=0),\\ \left\{\prod_{k=0}^{m-1}(1-8s(1+2k)^{-2})\right\}\Big/\{\cos(\pi\sqrt{2s})\} & (m=1,2,\ldots).\end{cases}$$

(Gradshteyn & Ryzhik, 1965, formula 1.431-3).

The $H(\frac{1}{2},0)$ distribution (the mixing distribution for the hyperbolic cosine distribution) appears as a first hitting time for a Brownian motion. See Theorem 4.1 below.

The $H(1,0)$ distribution (the mixing distribution for the logistic distribution) appears in several important contexts in statistics and probability. In particular, it is:

(a) the asymptotic distribution of $4D^2$, where D is the Kolmogorov–Smirnov goodness-of-fit statistic (see, for example, Durbin, 1973, p. 22);
(b) the asymptotic distribution of $4\pi^2U^2$, where U^2 is Watson's goodness-of-fit statistic for a test of uniformity on the circle (see, for example, Durbin, 1973, pp. 38–39);
(c) the asymptotic distribution of π^2/m^{*2}, where m^* is the Hodges–Ajne goodness-of-fit statistic for a test of uniformity on the circle (see, for example, Mardia, 1972, p. 185);
(d) a limiting first hitting time for Brownian motion; see Theorem 4.1 below.

THEOREM 4.1. *The $H(1,0)$ and $H(\frac{1}{2},0)$ distributions appear as first hitting time distributions for Brownian motion.*

Proof. The distribution $H(1,0)$ has moment generating function

$$\varphi(s)=\frac{\pi\sqrt{2s}}{\sin(\pi\sqrt{2s})}.$$

Since $u(x,s)=\{\sin(x\sqrt{2s})\}/\sqrt{2s}$ is the solution of (a prime denotes differentiation with respect to x) $\frac{1}{2}u''+su=0$, with boundary conditions $u(0,s)=0$ and $(\partial u/\partial x)(0,s)=1$, it follows that $\Phi(s)=\sin(a\sqrt{2s})/\sin(b\sqrt{2s})$ is the moment generating function of the first hitting time (τ_{ab}) to b in a standard Brownian motion with absorbing barrier at 0 and starting at a $(0<a<b)$. Since $\Phi(0)=a/b$, the function $(b/a)\Phi(s)$ is the moment generating function of τ_{ab} conditional on not hitting 0. Now, as $a\to 0$,

$$\frac{b}{a}\Phi(s)\to\frac{b\sqrt{2s}}{\sin(b\sqrt{2s})}$$

and if $b=\pi$ this is the moment generating function given above.

Similarly, the function $u(x,s)=\cos(x\sqrt{2s})$ is the solution of $\frac{1}{2}u''+su=0$, with the boundary conditions $u(0,s)=1$ and $(\partial u/\partial x)(0,s)=0$. Hence $\Psi(s)=\cos(a\sqrt{2s})/\cos(b\sqrt{2s})$ is the moment generating function of the first hitting time to b in a standard Brownian motion with a reflecting barrier at 0 and starting at a $(0<a<b)$. When $a=0$ and $b=\pi$ the moment generating function of $H(\frac{1}{2},0)$ is obtained.

The hitting times of Theorem 4.1 are studied in more detail by Kent (1980).

5 A Tauberian theorem

In this section we shall examine the relationship between the tail behaviour of a distribution function $F(du)$ on $(-\infty,\infty)$ and its moment generating function $\phi(s)=\int e^{su}F(du)$, where the integral is over $(-\infty, \infty)$. For simplicity we shall formulate our results in terms of the *right* tail of $F(du)$ but similar results can also be stated for the *left* tail. The results will be used in the next section to study the tail behaviour of normal variance-mean mixtures.

Define

$$\psi_- = \inf\{s \in R : \phi(s) < \infty\}, \quad \psi_+ = \sup\{s \in R : \phi(s) < \infty\}, \tag{5.1}$$

so that $-\infty \leqslant \psi_- \leqslant 0 \leqslant \psi_+ \leqslant \infty$. To get interesting results we shall suppose that

$$\psi_+ < \infty, \quad \psi_+ - \psi_- > 0. \tag{5.2}$$

Define

$$F_k(u) = \begin{cases} \int_{(0,u]} w^k e^{\psi_+ w} F(dw) & (u \geqslant 0), \\ 0 & (u \leqslant 0), \end{cases} \tag{5.3}$$

and let $\phi^{(k)}(s)$ denote the kth derivative of $\phi(s)$, where throughout this section k is assumed to be a nonnegative integer. If we partition $\phi(s) = \phi_1(s) + \phi_2(s)$, where

$$\phi_1(s) = \int_{(-\infty,0]} e^{su} F(du), \quad \phi_2(s) = \int_{(0,\infty)} e^{su} F(du),$$

then $\phi_2^{(k)}(s + \psi_+)$ is the moment generating function of $F_k(du)$.

Recall that a function $L(x)$ is said to be of *slow variation* as $x \to \infty$ if $L(tx)/L(x) \to 1$ as $x \to \infty$ for all $t > 0$. Examples include $L(x) = \text{constant}$ and $L(x) = \log x$. The following lemma which follows from Feller (1971, p. 282) will be useful.

LEMMA 5.1. *If $L(x)$ is a function of slow variation then for all $\varepsilon > 0$ there exists $A > 0$ such that for all $y \geqslant x \geqslant A$,*

$$(x/y)^\varepsilon \leqslant L(x)/L(y) \leqslant (y/x)^\varepsilon. \tag{5.4}$$

As a simple consequence we can also derive the following lemma (de Haan, 1970, p. 15).

LEMMA 5.2. *Let $L(x)$ be a function of slow variation, let $\rho \geqslant 0$ and let $c \geqslant 0$. Then as $x \to \infty$*

$$\int_c^x u^\rho L(u)\, du \sim (\rho + 1)^{-1} x^{\rho+1} L(x). \tag{5.5}$$

The following theorem relates the right-hand tail behaviour of $F(du)$ to the behaviour of $\phi(s)$ as $s \uparrow \psi_+$.

THEOREM 5.1. *Consider the asymptotic formulae*

$$F_k(u) \sim (\lambda + k)^{-1} u^{\lambda+k} L(u) \quad (u \to \infty), \tag*{$(5.6)_k$}$$

$$\phi^{(k)}(s + \psi_+) \sim \Gamma(\lambda + k)\, |s|^{-\lambda-k} L(1/|s|) \quad (s \uparrow 0), \tag*{$(5.7)_k$}$$

where $L(.)$ is a (fixed) function of slow variation. If either $(5.6)_k$ or $(5.7)_k$ holds for some $k > -\lambda$, then both $(5.6)_k$ and $(5.7)_k$ hold for all $k > -\lambda$.

Proof. Under assumptions (5.2) it is clear that $\phi_1^{(k)}(s + \psi_+)$ is bounded as $s \uparrow 0$ for all $k \geqslant 0$, so that a formula equivalent to $(5.7)_k$ can be obtained by replacing $\phi^{(k)}(s + \psi_+)$ by $\phi_2^{(k)}(s + \psi_+)$.

Thus this theorem is really only making an assertion about a measure on $(0, \infty)$.

By Feller's (1971, p. 445) Tauberian theorem for measures on $(0, \infty)$ it follows that $(5.6)_k$ holds if and only if $(5.7)_k$ holds.

Integrating $F_{k+1}(u)$ by parts and using (5.5) and $(5.6)_k$ yields

$$\begin{aligned} F_{k+1}(u) &= uF_k(u) - \int_{(0,u]} F_k(w)\,dw \\ &\sim \frac{u^{\lambda+k+1}L(u)}{(\lambda+k)} - \frac{u^{\lambda+k+1}L(u)}{(\lambda+k)(\lambda+k+1)} \\ &= (\lambda+k+1)^{-1}u^{\lambda+k+1}L(u) \end{aligned}$$

as $u \to \infty$. Hence $(5.6)_k$ implies $(5.6)_{k'}$ for all $k' \geq k$.

Similarly, integrating $\phi^{(k)}(s'+\psi_+)$ between $-c$ and s (where $0<c<\psi_+-\psi_-$ and $k-1>-\lambda$) and using (5.5) and $(5.7)_k$ yields

$$\begin{aligned} \phi^{(k-1)}(s+\psi_+) &= \phi^{(k-1)}(-c+\psi_+) + \int_{-c}^{s} \phi^{(k)}(s'+\psi_+)\,ds' \\ &\sim \text{const} + \frac{\Gamma(\lambda+k)}{\lambda+k-1}|s|^{1-\lambda-k}L(1/|s|) \\ &\sim \Gamma(\lambda+k-1)\,|s|^{1-\lambda-k}L(1/|s|) \end{aligned}$$

as $s \uparrow 0$. Hence $(5.7)_k$ implies $(5.7)_{k'}$ for all k' satisfying $-\lambda<k'<k$. Thus the theorem follows.

The above theorem is stated in terms of the (modified) *distribution function* F_k but under mild conditions, it can be extended to give information about the *density* $f(u)$ of F.

THEOREM 5.2 (*Feller*, 1971, *p.* 466). *If* $F_k(u)$ *has an ultimately monotone derivative for some* $k>-\lambda$ *and if* $(5.6)_k$ *or* $(5.7)_k$ *holds, then* F *has a density* $f(u)$ *satisfying*

$$f(u) \sim e^{-\psi_+ u}u^{\lambda-1}L(u) \quad (u \to \infty). \tag{5.8}$$

Conversely, if (5.8) *holds with* L *a function of slow variation* (*whether or not* $e^{\psi_+ u}u^{\lambda+k}f(u)$ *is ultimately monotone for any* $k>-\lambda$), *then* $(5.6)_k$ *and* $(5.7)_k$ *hold for all* $k>-\lambda$.

Remark 1. The value of λ appearing in Theorem 5.1 is of course unique, and the function $L(u)$ is unique up to any asymptotically equivalent function.

Remark 2. The classification in Theorem 5.1 is *not* exhaustive. For example, if $f(u) \sim \exp(-u^{\frac{1}{2}})$ then $\psi_+=0$, but $(5.6)_k$ does not hold for any finite value of λ. However, the description in Theorem 5.1 is adequate for most purposes, and in particular $L(u)$ is usually asymptotically equal to a constant.

Remark 3. If $\psi_+>0$ then λ may take any real value in Theorem 5.1. If $\psi_+=0$ then necessarily $\lambda \leq 0$ in Theorem 5.1. Further, if $\psi_+=0$ and $\lambda=0$ in Theorem 5.2, then $L(u)$ must be such that

$$\int_{u_0}^{\infty} u^{-1}L(u)\,du < \infty,$$

for some $u_0>0$.

Remark 4. The conditions of Theorem 5.2 seem very mild but might be difficult to check in practice.

Remark 5. An important special case occurs when $F(du)$ is concentrated on $[0, \infty)$. Then clearly $\psi_- = -\infty$.

6. Tail behaviour of normal variance-mean mixtures

In this section we shall study the way in which the tail behaviour of a normal variance-mean mixture depends on the tail behaviour of the mixing distribution, in the one-dimensional case.

Let $F(du) = f(u)\,du$ denote the mixing distribution on $[0, \infty)$ with moment generating function $\varphi(s)$. Consider first the normal variance mixture

$$g(x) = \int_0^\infty e^{-x^2/2u}(2\pi u)^{-\frac{1}{2}} f(u)\,du. \tag{6.1}$$

THEOREM 6.1. *Suppose g is a normal variance mixture and that the tail of the mixing distribution $f(u)$ satisfies (5.8). If $\psi_+ = 0$, then*

$$g(x) \sim (2\pi)^{-\frac{1}{2}} 2^{\frac{1}{2}-\lambda} \Gamma(\tfrac{1}{2}-\lambda)\,|x|^{2\lambda-1} L(x^2), \quad |x| \to \infty \tag{6.2}$$

and if $\psi_+ > 0$, then

$$g(x) \sim (2\psi_+)^{-\frac{1}{2}\lambda} L(|x|)\,|x|^{\lambda-1} \exp\left(-(2\psi_+)^{\frac{1}{2}}|x|\right), \quad |x| \to \infty. \tag{6.3}$$

Proof. Throughout this proof write $\psi_+ = \psi$ for simplicity. To illustrate the technique let us first consider the case $L(u) \equiv 1$. (Note that if $\lambda < 0$ then $f(u)$ is not integrable near $u = 0$ but the integral (6.1) is still finite for $x \neq 0$). If $\psi = 0$ then the change of variable $u = x^2/v$ reduces (4.1) to a gamma integral from which (6.2) follows immediately. If $\psi > 0$, then with the change of variable $u = (2\psi/x^2)^{\frac{1}{2}} e^t$, equation (6.1) can be expressed as a modified Bessel function with asymptotic behaviour

$$\begin{aligned} g(x) &= (2/\pi)^{\frac{1}{2}} (x^2/2\psi)^{\frac{1}{2}\lambda - \frac{1}{4}} K_{\lambda-\frac{1}{2}}((2\psi)^{\frac{1}{2}} x) \\ &\sim (2\psi)^{-\frac{1}{2}\lambda} x^{\lambda-1} \exp\{-(2\psi)^{\frac{1}{2}} x\} \end{aligned}$$

(Abramowitz & Stegun, 1972, formulae 9.6.24, 9.7.2), from which (6.3) follows.

For a mixing density $f(u)$ with tail behaviour determined by a general slowly varying function $L(u)$, the theorem can be deduced by using the above argument together with (5.4) and the following observations.

(a) The tail behaviour of $g(x)$ does not depend on the behaviour of $L(u)$ on any finite interval $(0, c)$.

(b) The asymptotic formula $K_\rho(x) \sim (\pi/2x)^{\frac{1}{2}} e^{-x}$ for large x holds *uniformly* over compact sets of ρ.

The tail behaviour of an arbitrary normal variance-mean mixture may be deduced from the remark in § 2 on the exponential family generated by a normal variance mixture and Theorem 6.1, in the following way. Suppose θ satisfies $\varphi(\frac{1}{2}\theta^2) < \infty$ (so that, in particular, $|\theta| \leq (2\psi_+)^{\frac{1}{2}}$, where ψ_+ is given by (5.1)). With $g(x)$ given by (6.1) define

$$g_\theta(x) = e^{\theta x} g(x)/\varphi(\tfrac{1}{2}\theta^2). \tag{6.4}$$

This is the probability density function of the normal variance-mean mixture with drift θ and mixing density

$$f_\gamma(u) = e^{\gamma u} f(u)/\varphi(\gamma); \tag{6.5}$$

where $\gamma = \frac{1}{2}\theta^2$. If $f(u)$ has tail of the form (5.8) then

$$f_\gamma(u) \sim e^{-\psi' u} u^{\lambda-1} L'(u), \tag{6.6}$$

where

$$\psi' = \psi_+ - \gamma = \psi_+ - \tfrac{1}{2}\theta^2, \quad L'(u) = L(u)/\varphi(\gamma). \tag{6.7}$$

If we start with a knowledge of $f(u)$ or $\varphi(s)$ then we can determine the tail behaviour of $g(x)$ from Theorem 6.1, and the tail behaviour of $g_\theta(x)$ is then found immediately using (6.4).

The tail behaviour of $g_\theta(x)$ can also be studied in terms of its own moment generating function $M_\theta(t)$ by using Theorems 5.1 and 5.2. In particular, the reader may check that if the mixing density $f(u)$ satisfies (5.8) so that $\varphi(s)$ satisfies $(5.7)_k$ as $s \uparrow \psi_+$ (with $\psi_+ > 0$) then the behaviour of $M(t)$ as $t \uparrow (2\psi_+)^{\frac{1}{2}}$ is compatible with the result of Theorem 6.1.

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Résumé

Nous presentons une vue d'ensemble des propriétes des lois probabilités qui sont mixtures 'variance-moyen' des distributions normale. Des resultats nouveaux sont inclus. En particulier, on discute les 'distributions z', c.-a.-d. les distributions produit par une transformation logistique des distributions Beta.

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