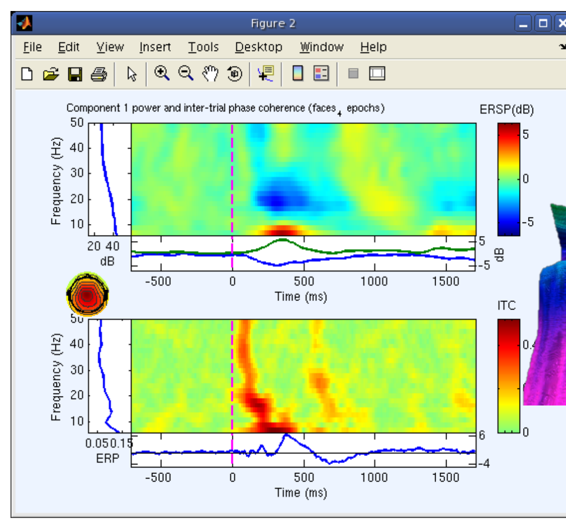
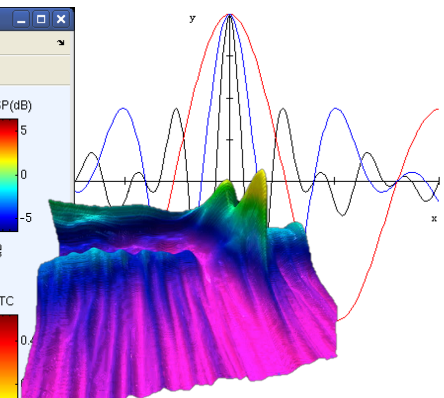
# Time-Frequency Analysis of Biophysical Time series

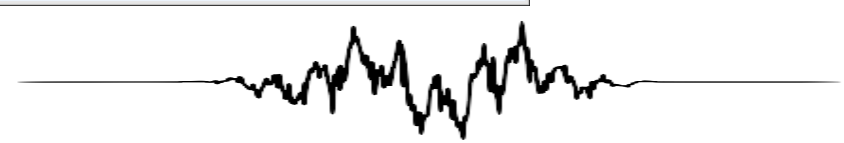

**Tim Mullen**

(with contributions from Arnaud Delorme)

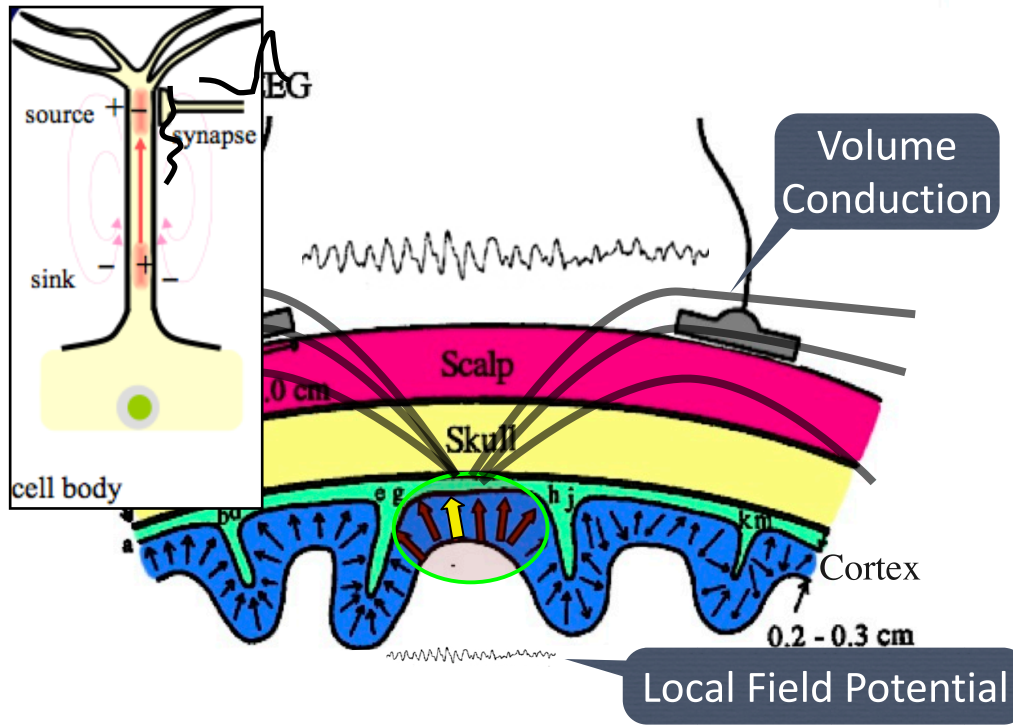
$$S(f) = \frac{1}{N}\sum_{t=0}^{N-1} x(t)e^{-2\pi i f t/N}$$



EEGLAB Workshop
Aspet, France 2017

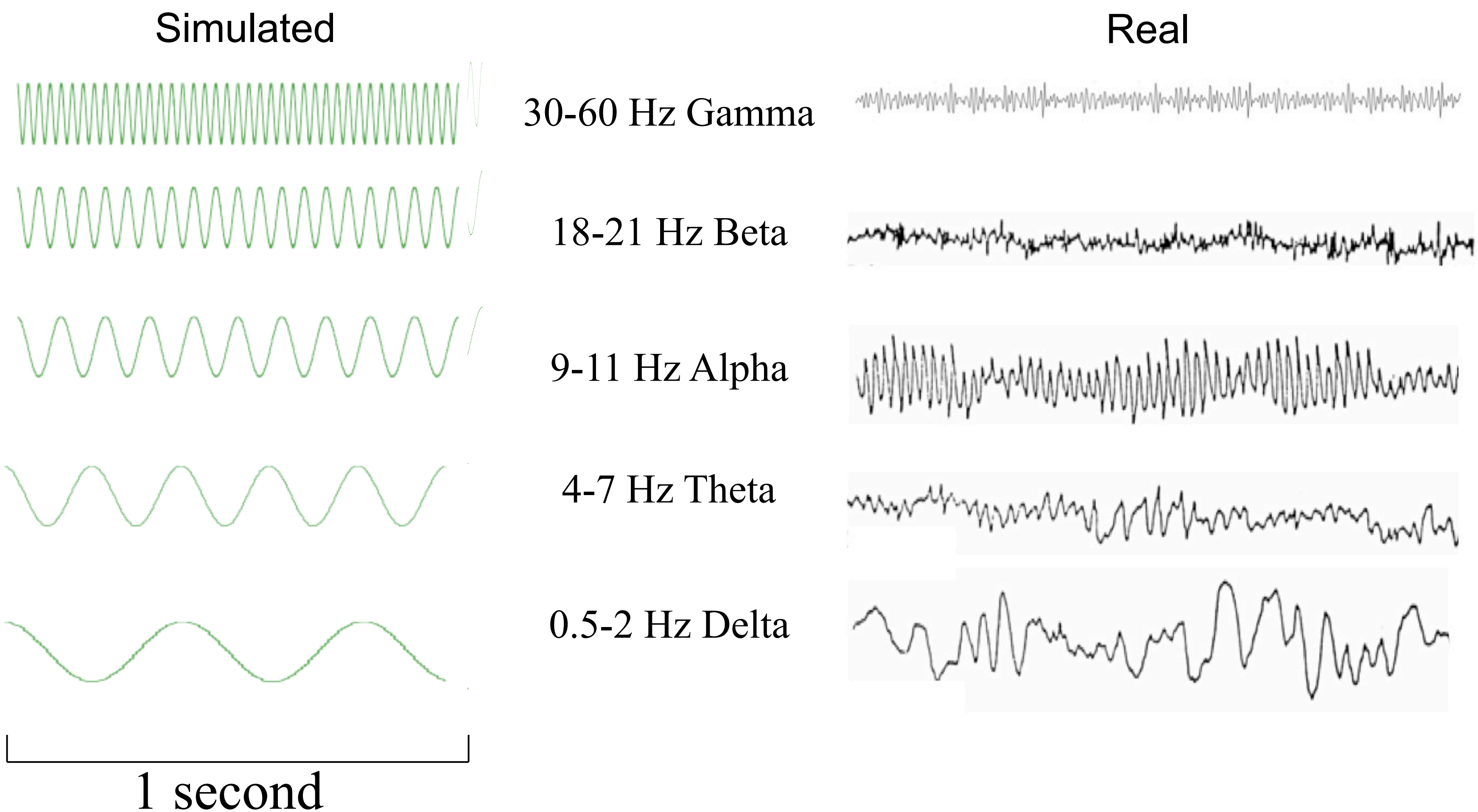
# Biophysics of EEG

Hans Berger (1873-1971)




Synchronicity of cell excitation (due to recurrent cortico-cortical and cortico-thalamo-cortical projections) determines amplitude and rhythm of the EEG signal

# Common Oscillatory Modes in EEG



# Sinusoids

phase shift

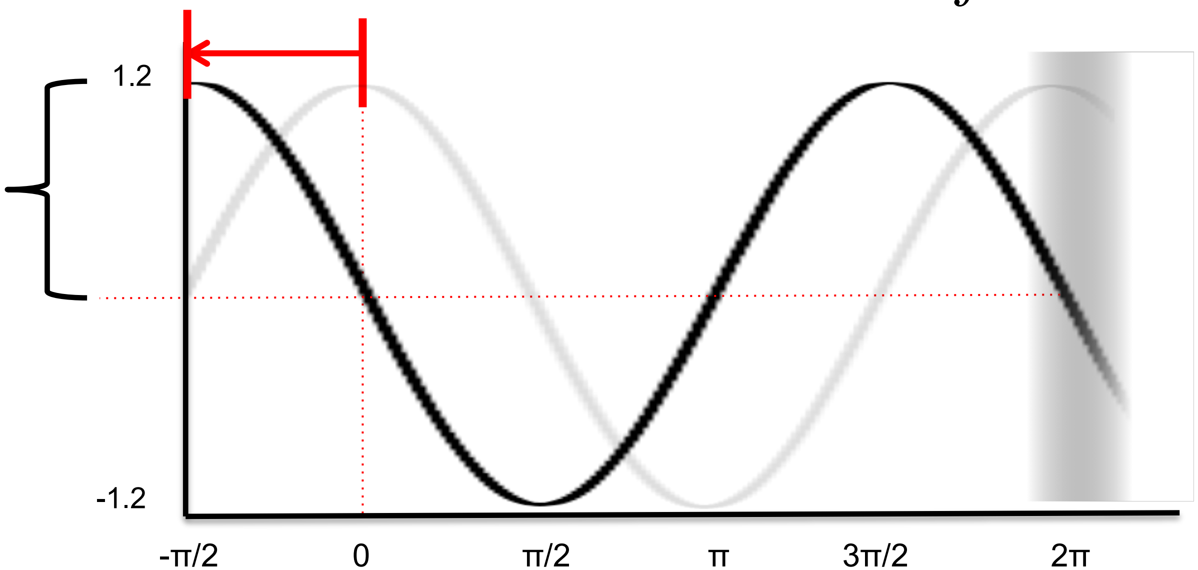
$$\theta = \pi / 2$$

angular frequency

$$\omega = 2\pi f = 2\pi \text{ rad/sec}$$

Amplitude

$A$=1.2



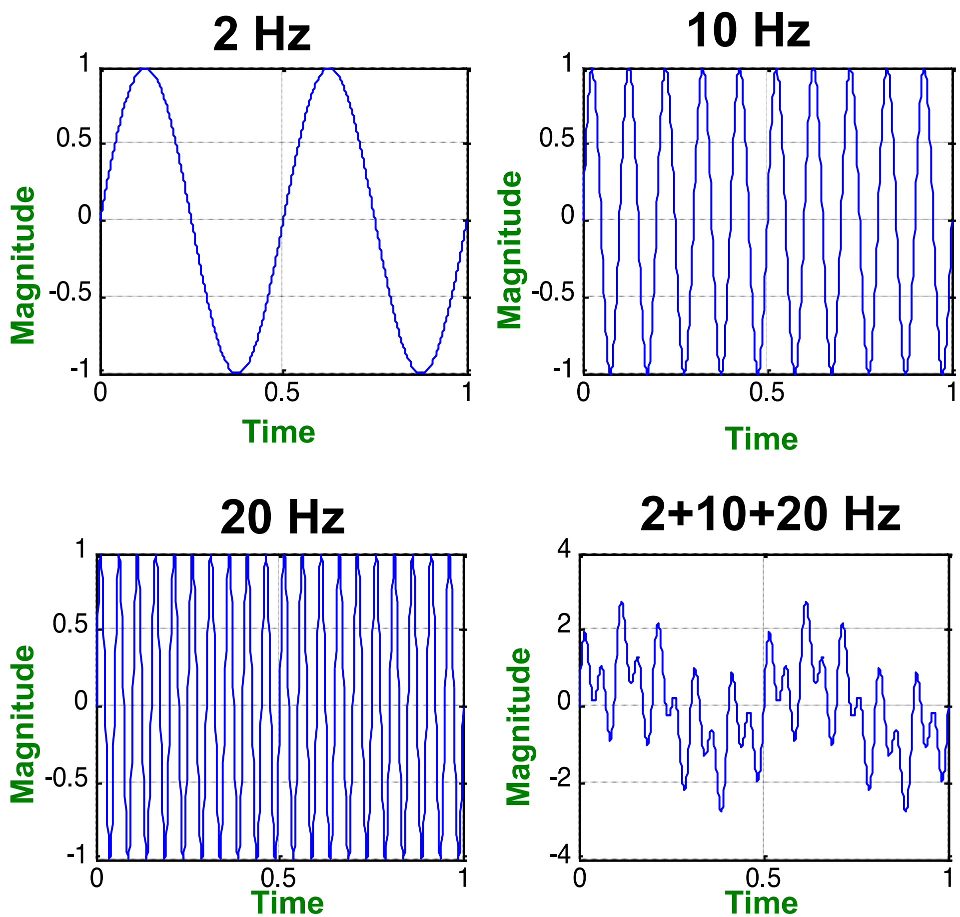

*t*

# Wide-sense stationary signals

The first and second moments (mean and variance) of the data distribution do not depend on time.

# Wide-sense stationary signals

Cyclostationary signals



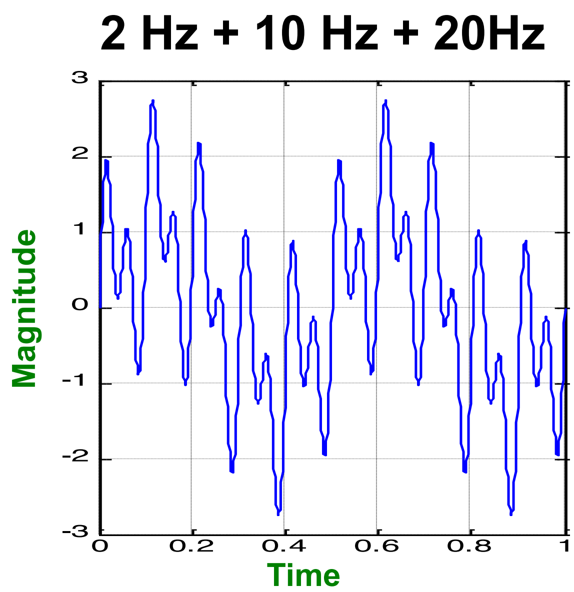

**Wide-sense Stationary**

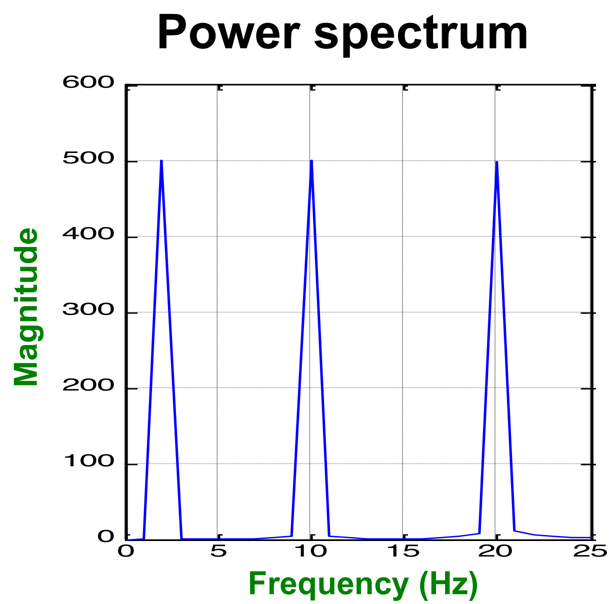
Slide courtesy of Petros Xanthopoulos, Univ. of Florida

# Wide-sense stationary signals

**Wide-sense**

**Stationary**




By looking at the power spectrum of the signal we can observe three frequency components (at 2Hz, 10Hz, and 20Hz respectively).

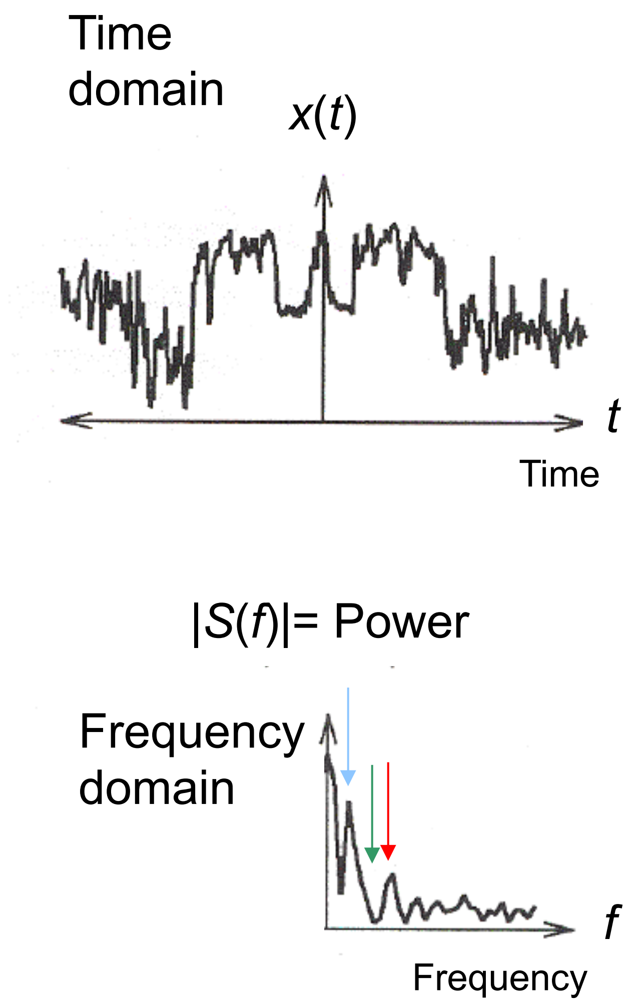
Slide courtesy of Petros Xanthopoulos, Univ. of Florida

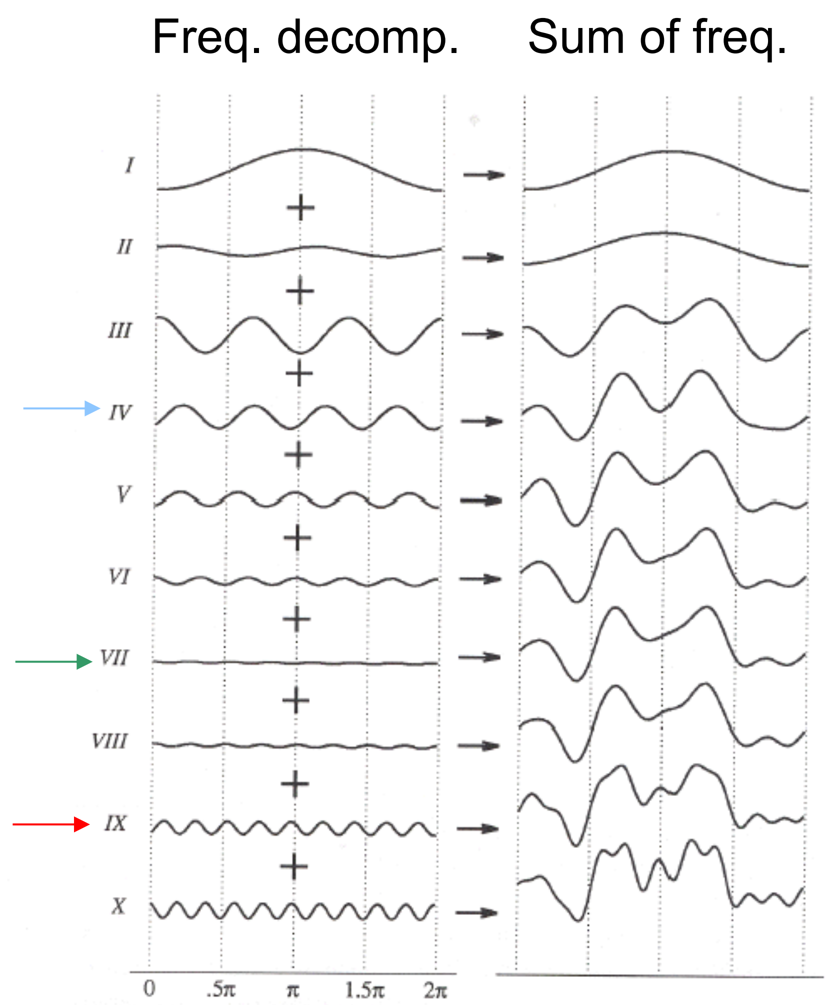
# Fourier’s Theorem

Any stationary, continuous process can be exactly described by an infinite sum of sinusoids of different amplitudes and phases.

Jean Baptiste Joseph Fourier
(1768 –1830)

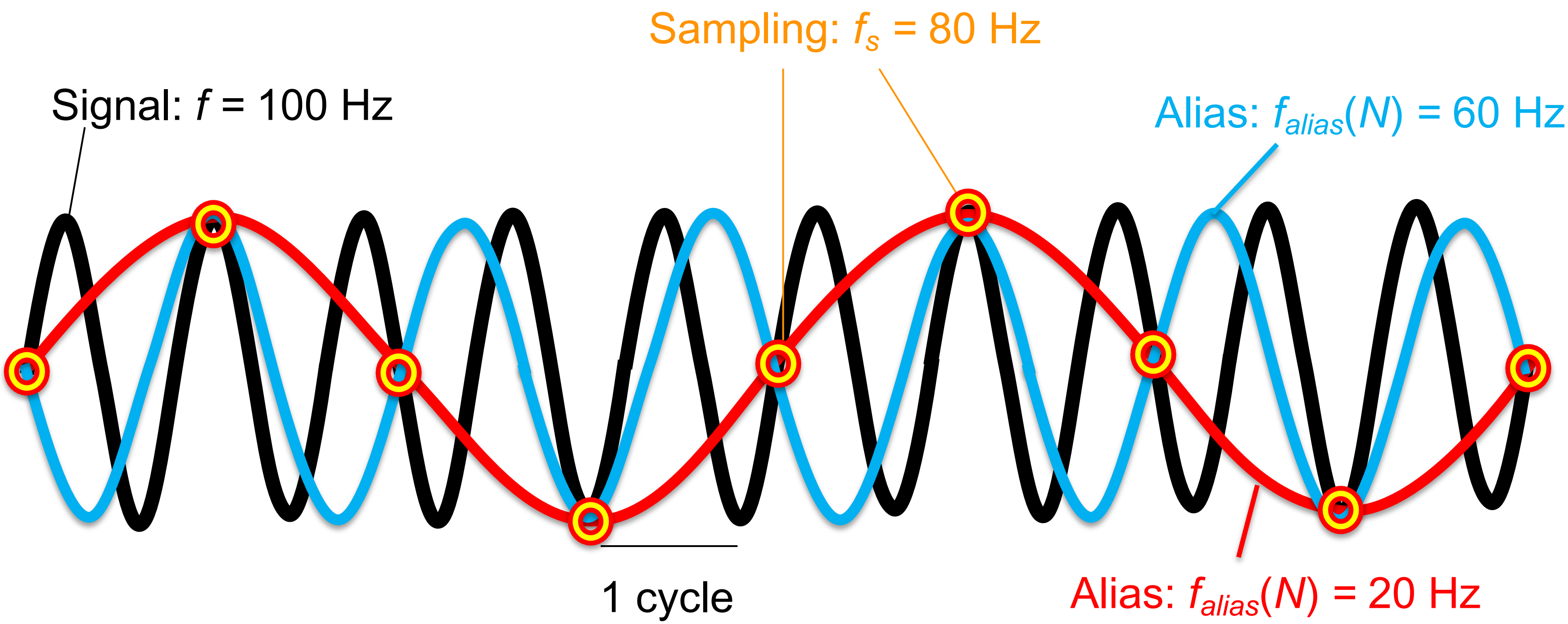
# Fourier Analysis




Figure, courtesy of Ravi Ramamoorthi & Wolberg

# Aliasing and the Nyquist Frequency




**Nyquist Frequency:**

The maximum frequency that can be uniquely recovered at a sampling rate of $f_s$

$$f_N = f_s / 2$$

Quiz: What should $f_N$ be for $f$ =100Hz?

$$f_{alias}(N) = |f - Nf_s|$$

$$f_s = \text{sampling rate}$$

Quiz: What is $N$ in the example above?

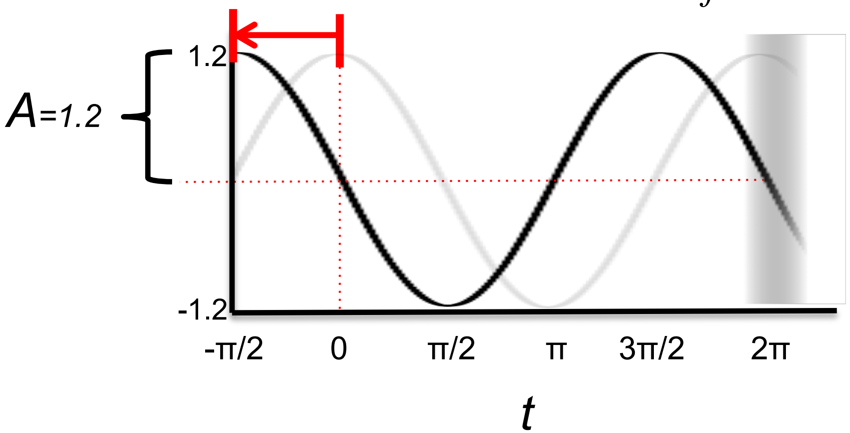
# Euler's Formula

phase shift

$\theta = \pi / 2$

angular frequency

$\omega = 2\pi f = 2\pi \text{ rad/sec}$




any real-valued sinusoid can be expressed as the sum of two complex numbers...

$$A \cdot \cos(\omega t + \theta) = \frac{A}{2} e^{i(\omega t + \theta)} + \frac{A}{2} e^{-i(\omega t + \theta)}$$

$$= \mathrm{Re}\{\underbrace{Ae^{i(\omega t + \theta)}}_{\textit{phasor}}\} = \mathrm{Re}\{S(\omega, t)\}$$

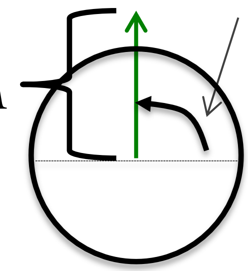
Amplitude

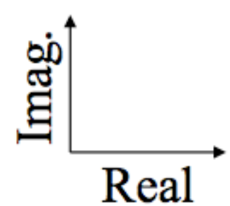
$|S(\omega, t)| = A$

Phase

$\theta = \angle S(\omega, t)$

$= \pi / 2$





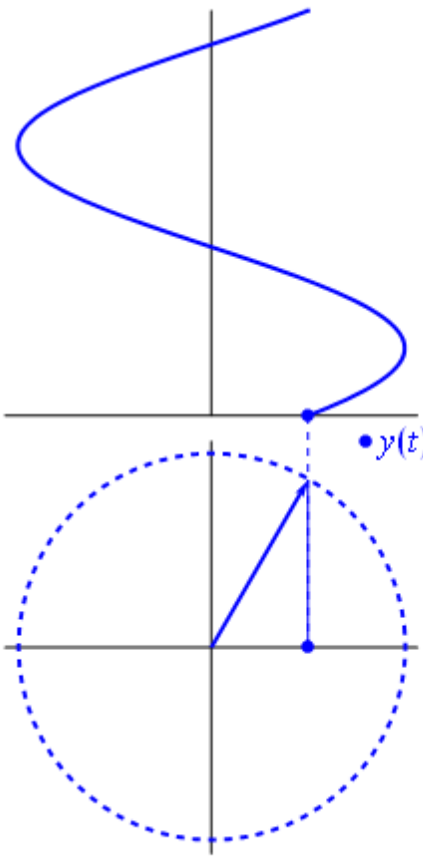

**Phasor**

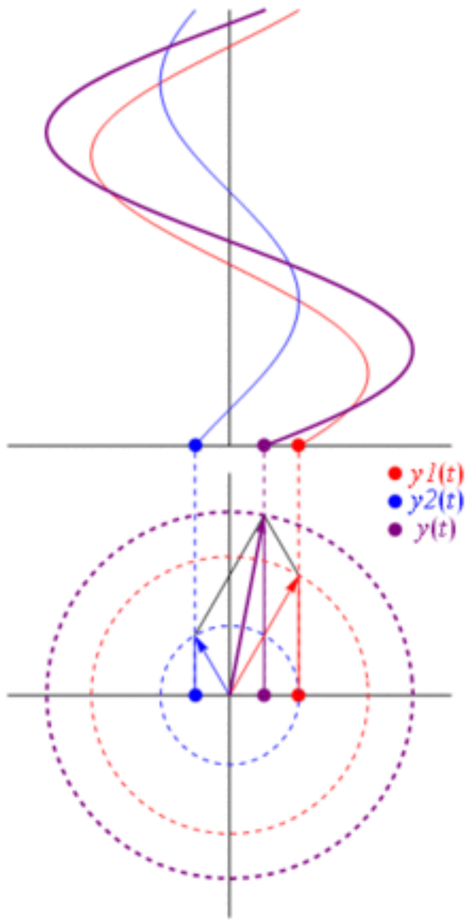
$Ae^{i(\omega t + \theta)}$

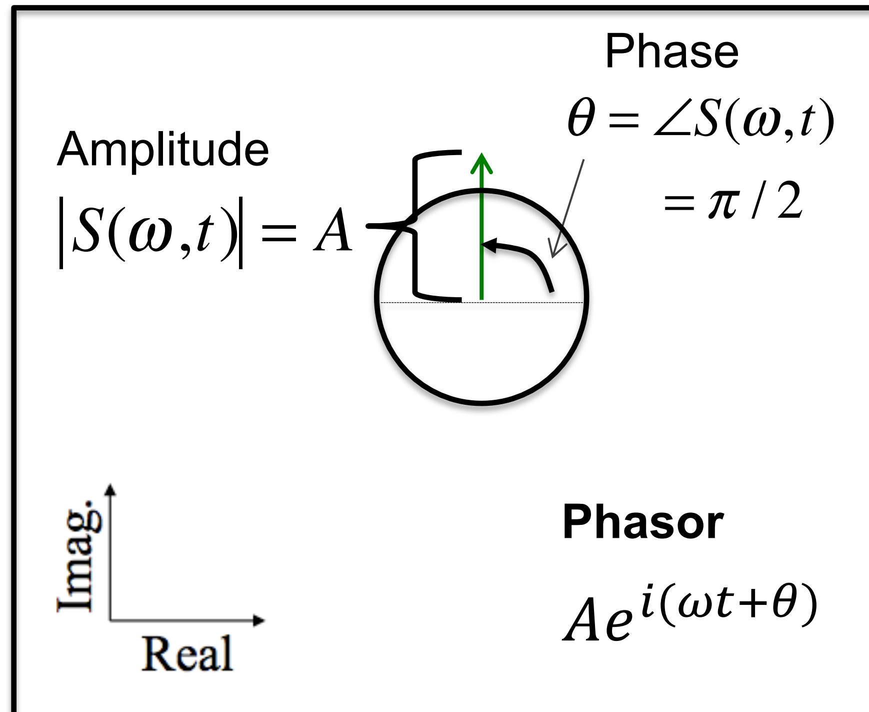
Shorthand phasor notation: $Ae^{i\theta}$

# Phasors

Rotation velocity (Rad/S; Hz)
= (angular) frequency (*w*; *f*)




animations courtesy Wikipedia

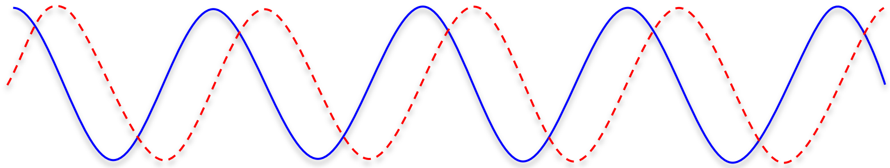
# Euler's Formula

Another view: a phasor as a complex sinusoid

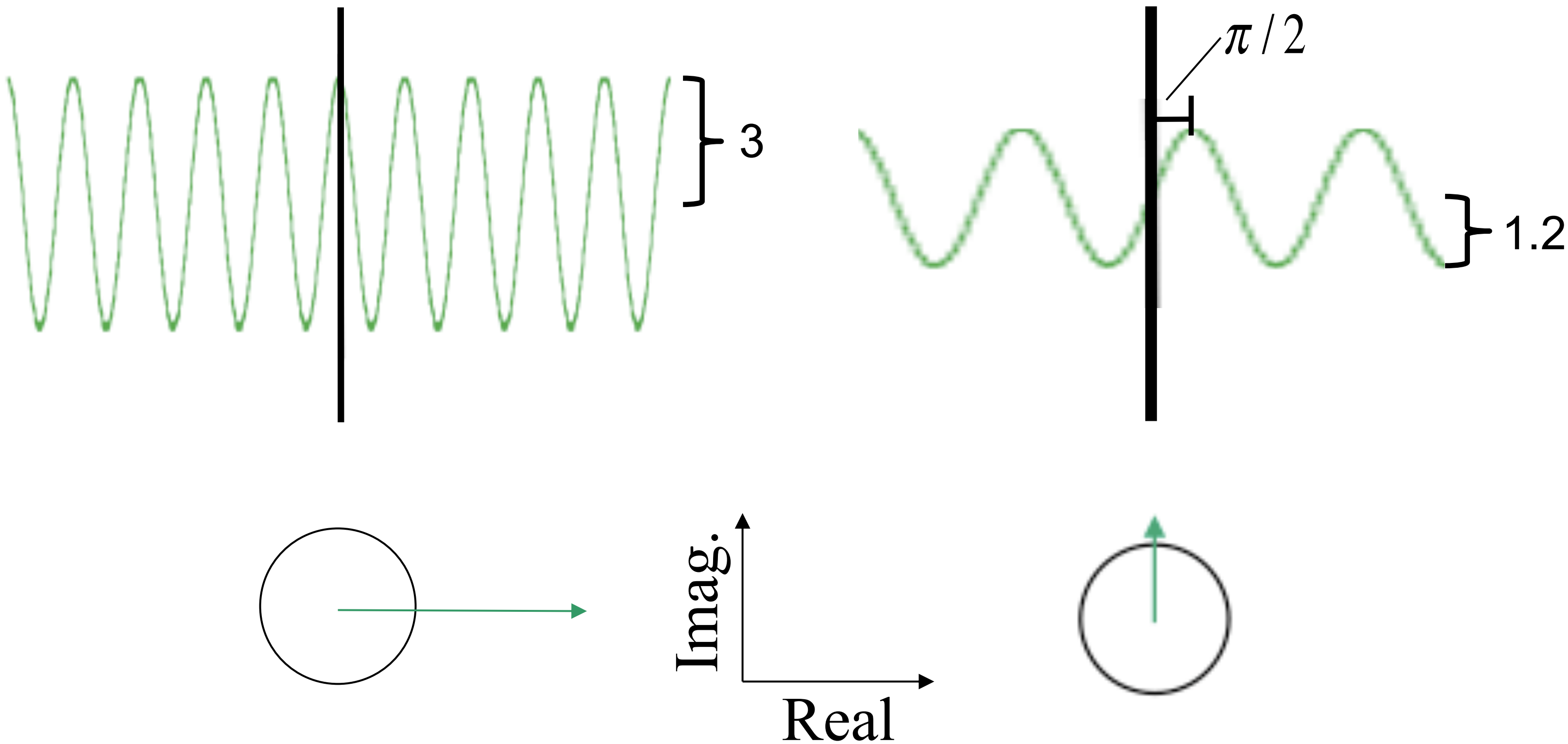
$$e^{i(\omega t+\theta)} = \cos(\omega t + \theta) + i\sin(\omega t + \theta)$$

Real part
Cosine component

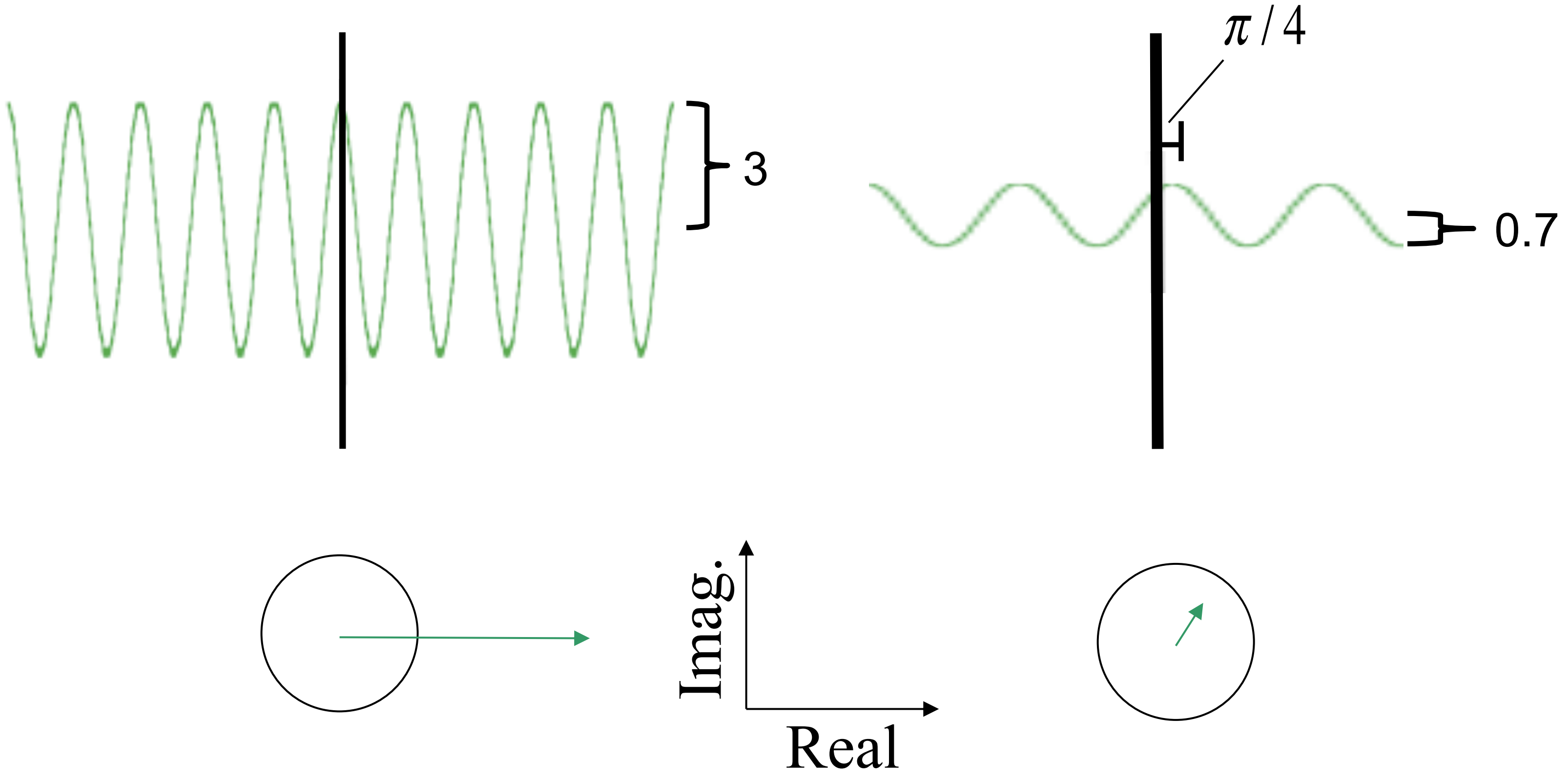
Imaginary part
Sine component



# Phasors: Example



# Phasors: Example



# Discrete Fourier Transform

Time → Frequency

**Forward transform**

$$S(f) = \frac{1}{N}\sum_{t=0}^{N-1} x(t)e^{-2\pi i f t/N}$$

Frequency → Time

**Inverse transform**

$$x(t) = \frac{1}{N}\sum_{f=0}^{N-1} S(f)e^{2\pi i f t/N}$$

N = number of samples

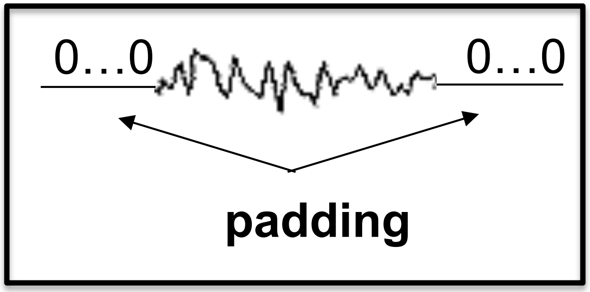
**Fast Fourier Transform (FFT)**

$$e^{\pm(2\pi i f t+\theta)} = \cos(2\pi f t+\theta) \pm i\sin(2\pi f t+\theta)$$

Power reflects the **covariance** between the original signal and a complex sinusoid at frequency *f*. Or you can think of it as the proportion of the signal variance explained by a sinusoid at frequency *f*

# Zero-padding

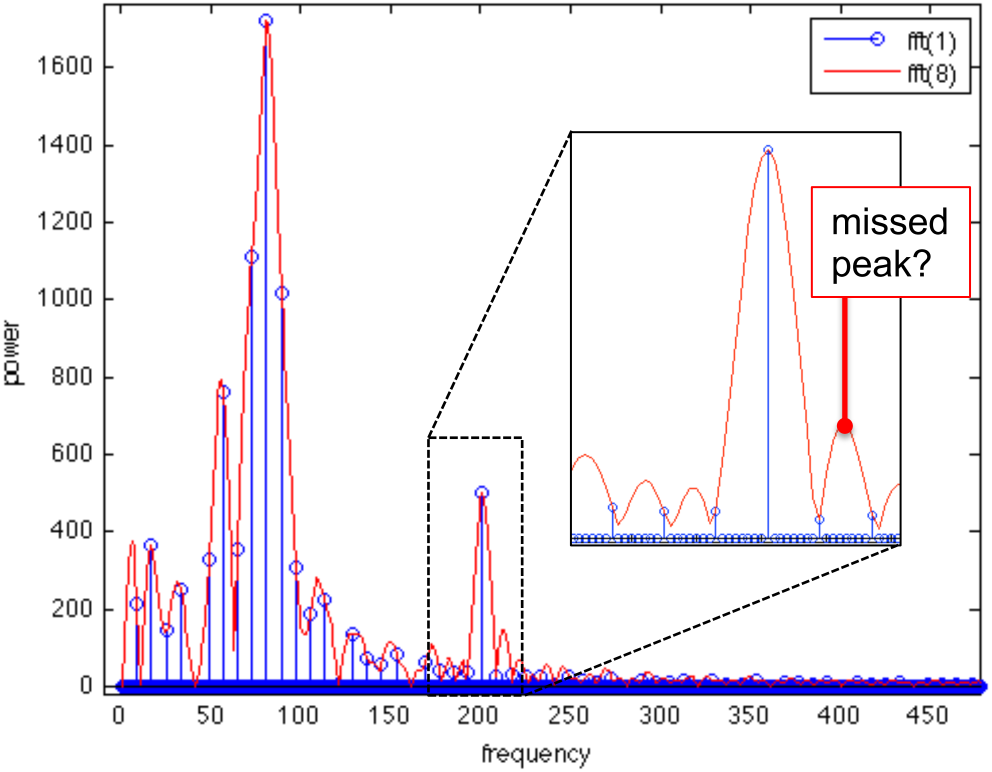
The DFT/FFT of a sequence of length N produces power estimates at N frequencies evenly distributed between 0 and the sampling rate (Fs), or floor(N/2+1) frequencies between 0 and the Nyquist rate, Fn=Fs/2.




Padding the signal with Q zeros achieves the following:

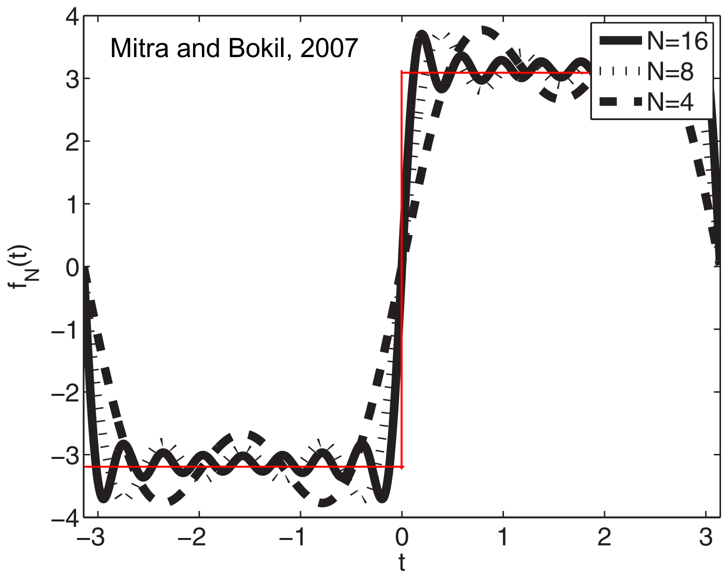
1) Allows enforcement of signal length as a power of two enabling FFT
2) Produces a smoother spectrum by increasing the number of frequency bins between 0 and Fs from N to N+Q (intermediate points are sinc interpolates)

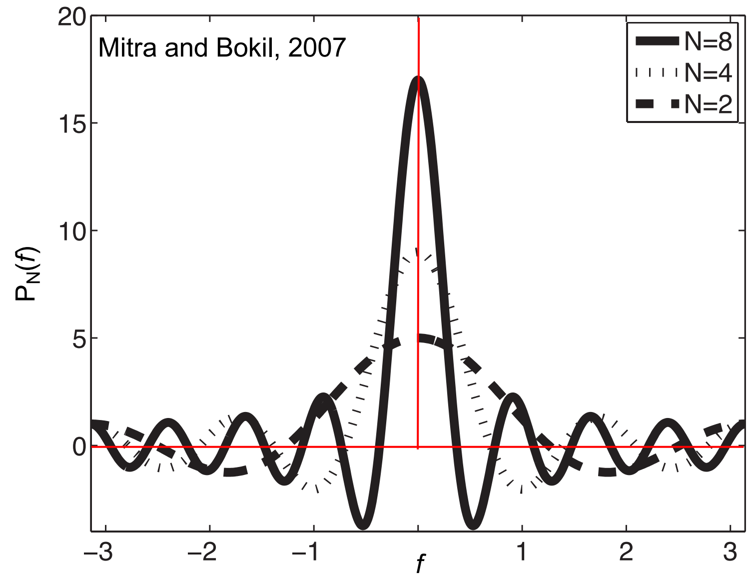
**Zero-padding does not increase frequency resolution (number of independent degrees of freedom)**



# Tapering

Fourier's Theorem lets us exactly represent any length *N*, continuous, stationary signal using a weighted sum of *N* sinusoids. Discontinuous functions must be approximated.




**Gibbs Phenomenon**

"Rippling" effect due to discontinuities in signal (e.g. edges of the truncated signal)

- Infinite number of frequencies required to approximate discontinuities
- This means infinite (or very large) number of samples required (not possible)

**What can we do?**

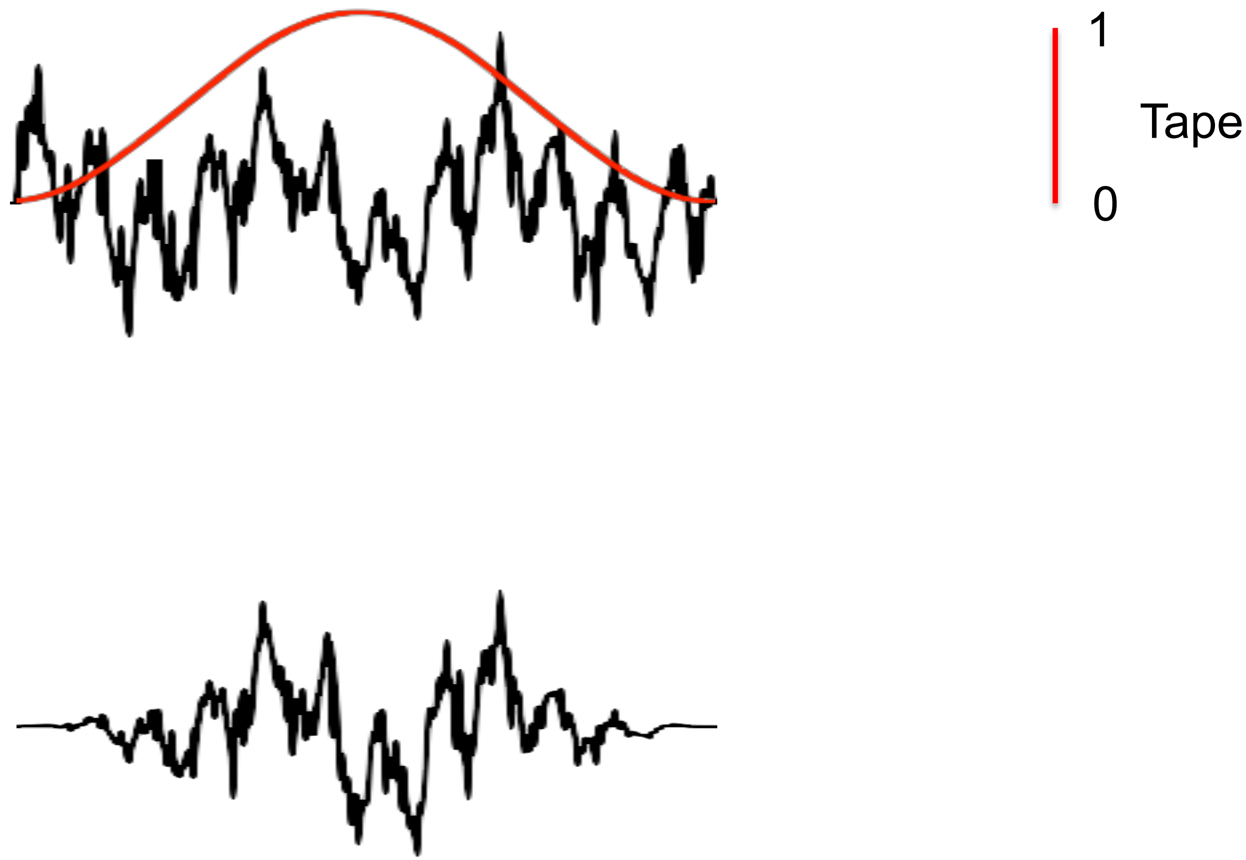
# Tapering

Smoothly decay signal to zero at endpoints to smooth discontinuity

EEG

X

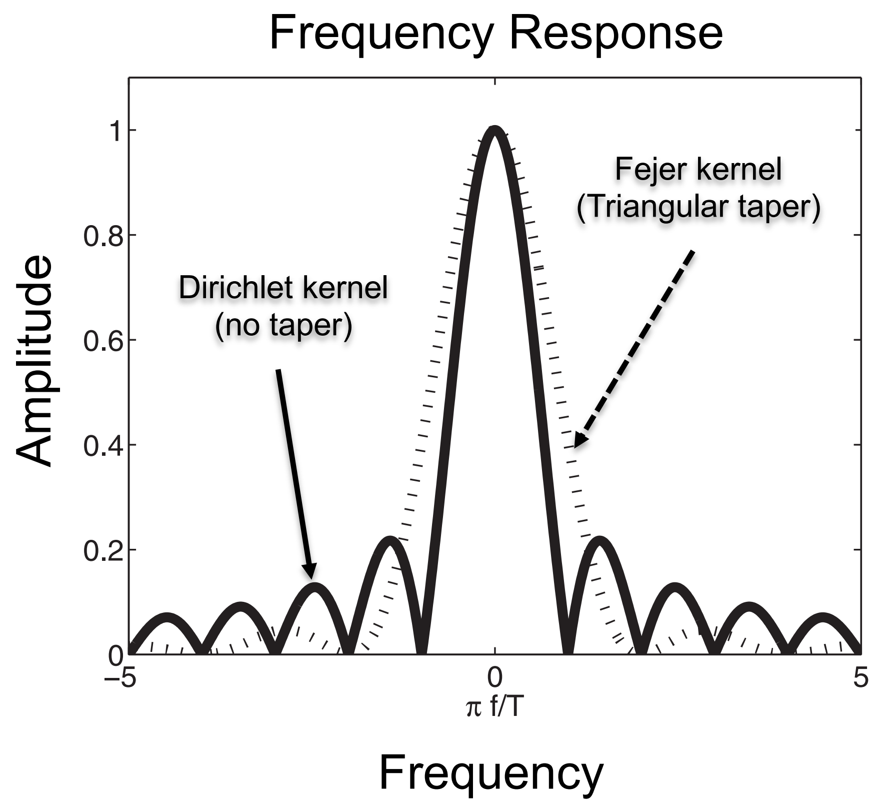
Tapered EEG



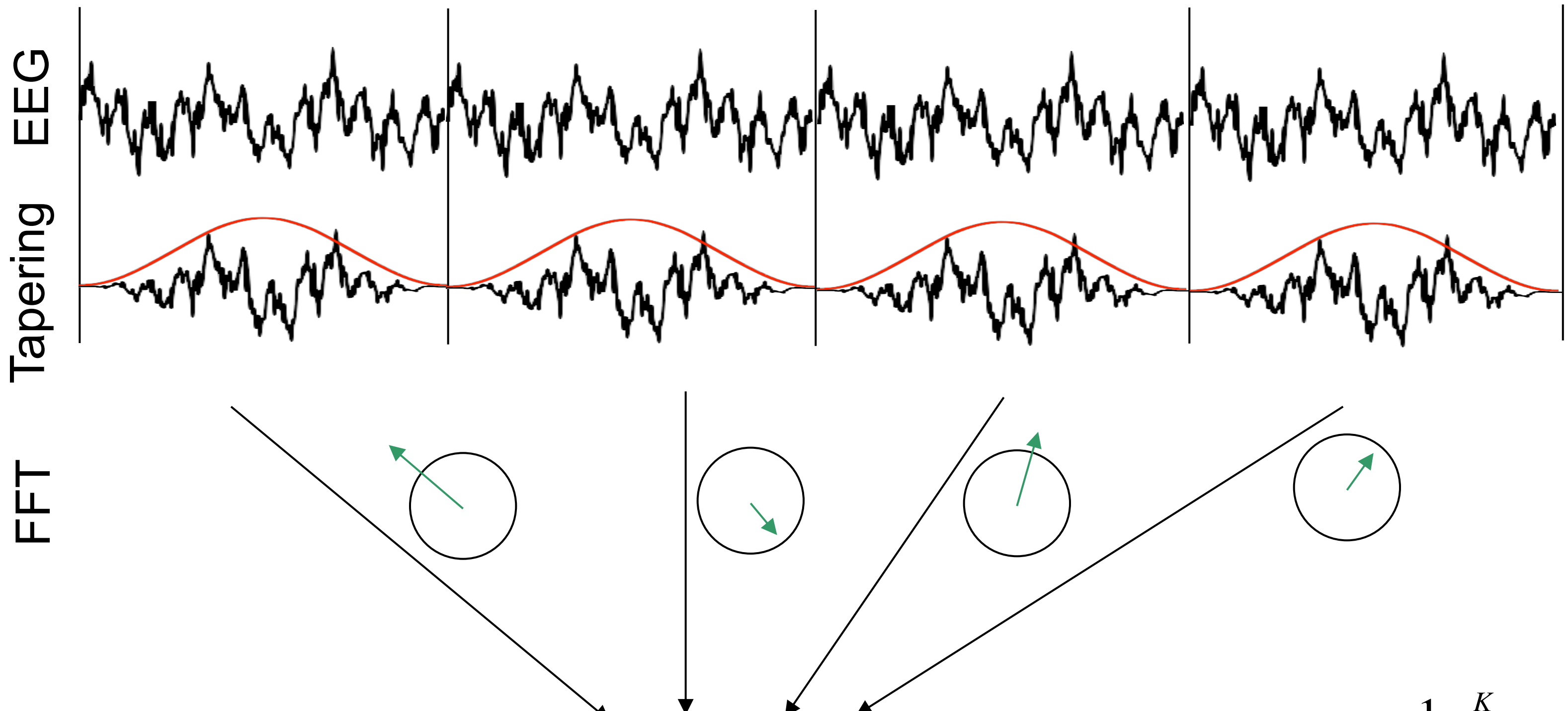
# Tapering

Tapering reduces the effect of the Gibbs phenomenon making it easier to identify "true" peaks in the spectrum from spurious ripple peaks (minimized broadband bias or "spectral leakage")

The cost is increased width of central peak (narrowband bias).

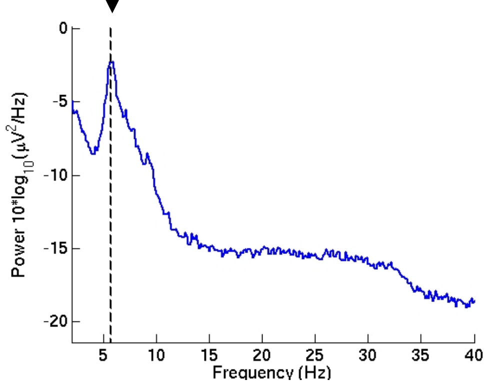


# Spectral Estimation via Welch's Method




Average of squared absolute values

$$S_{Welch}(f) = \frac{1}{K}\sum_{k=1}^{K}\left|S_k(f)\right|$$

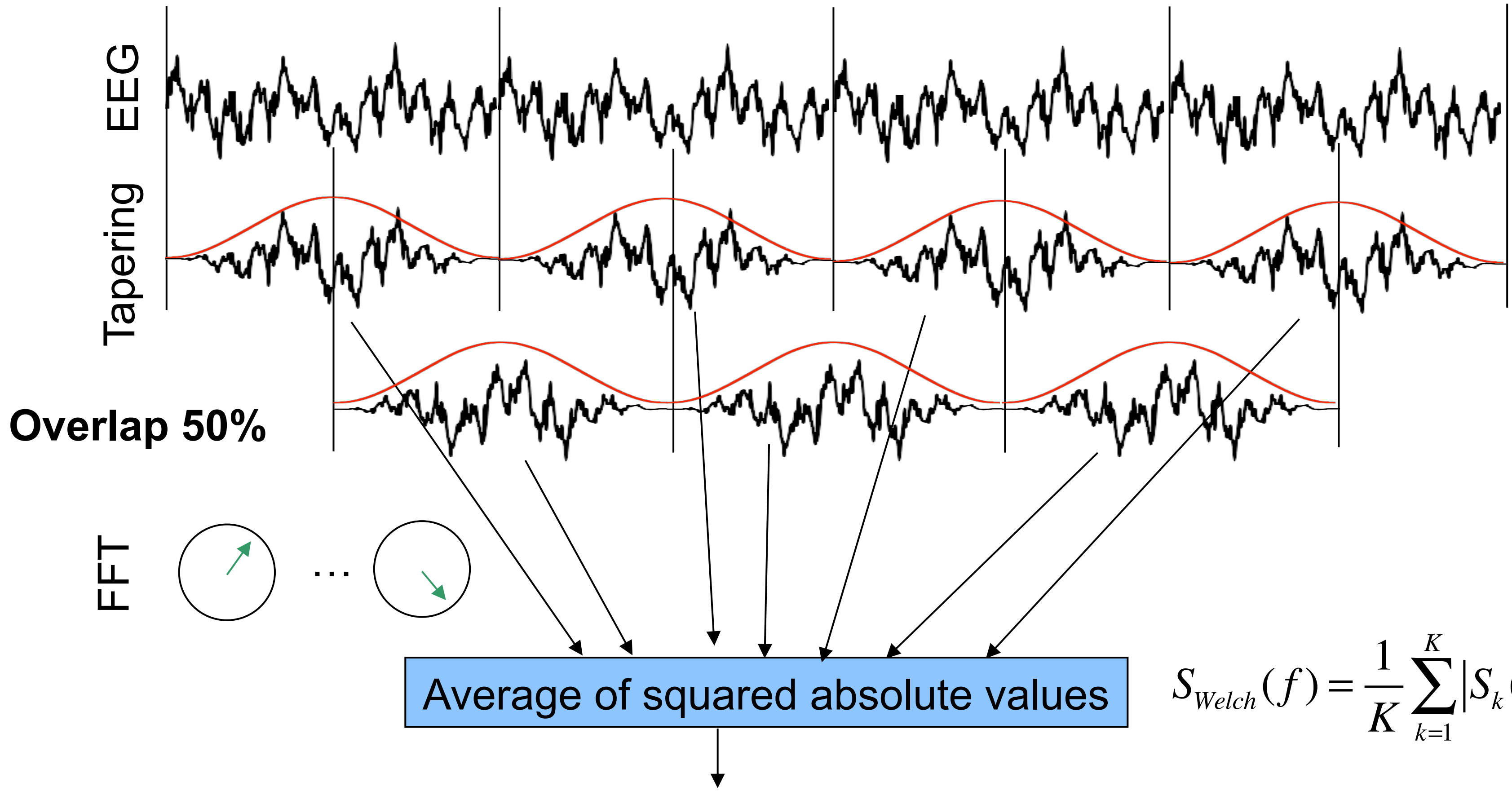



Given *K* windows:

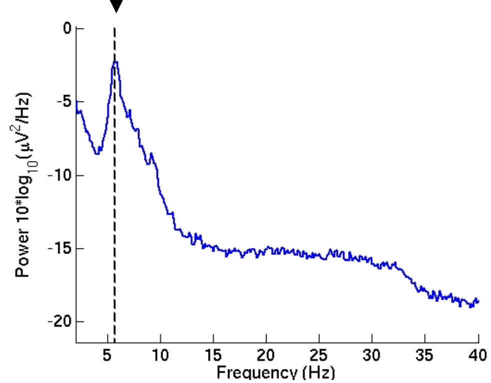
- Variance is reduced by a factor of *K*
- Frequency resolution also reduced by a factor of *K*

- Tapering also results in data loss → decreased frequency resolution (increased narrowband bias)
- Can we mitigate data loss?

# Spectral Estimation via Welch's Method



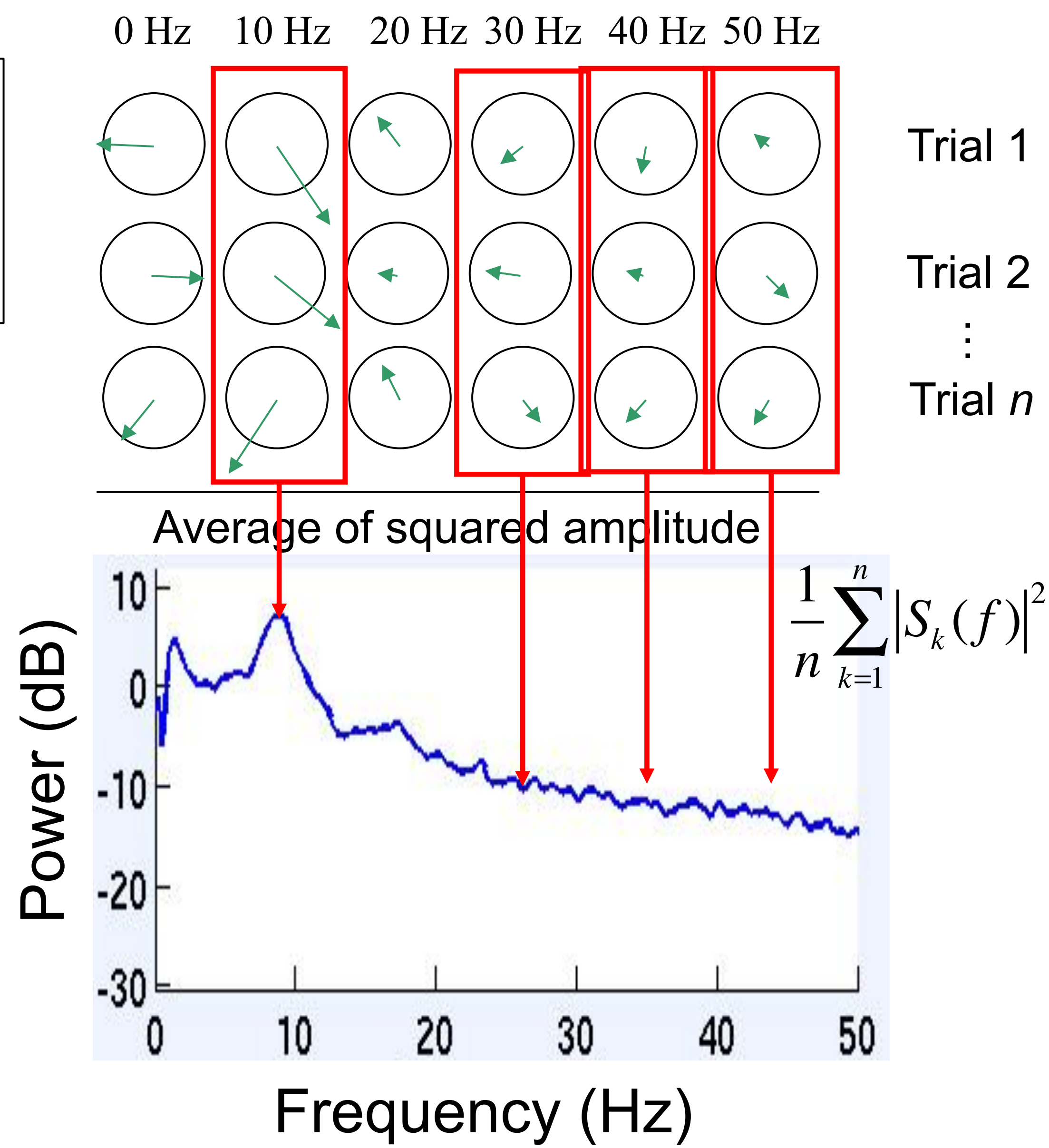

$$S_{Welch}(f) = \frac{1}{K}\sum_{k=1}^{K}\left|S_k(f)\right|$$




Overlap-add method results in reduced data loss due to tapering, while preserving spectral variance reduction ☺

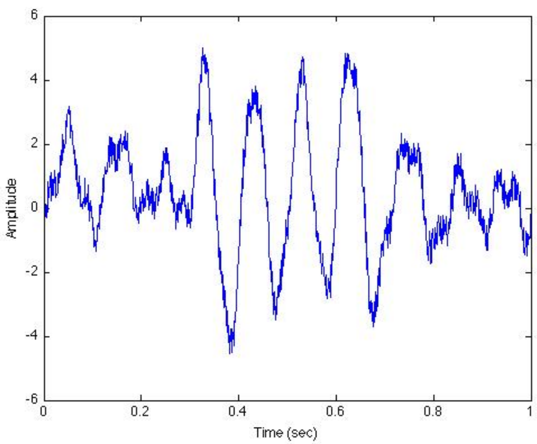
# Trial Averaging

Averaging spectra over *n* independent trials leads to further reduction of variance by a factor of *n*

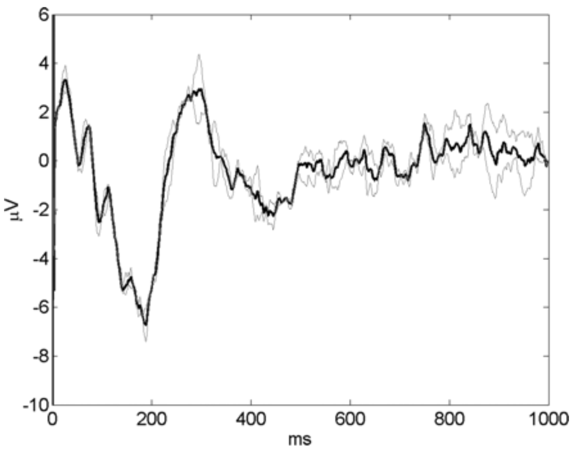


# Non-Stationary Signals

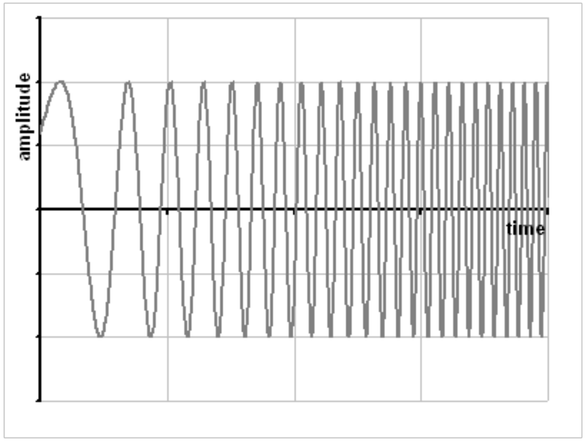
Non-stationary signals include bursts, chirps, evoked potentials, …




Burst

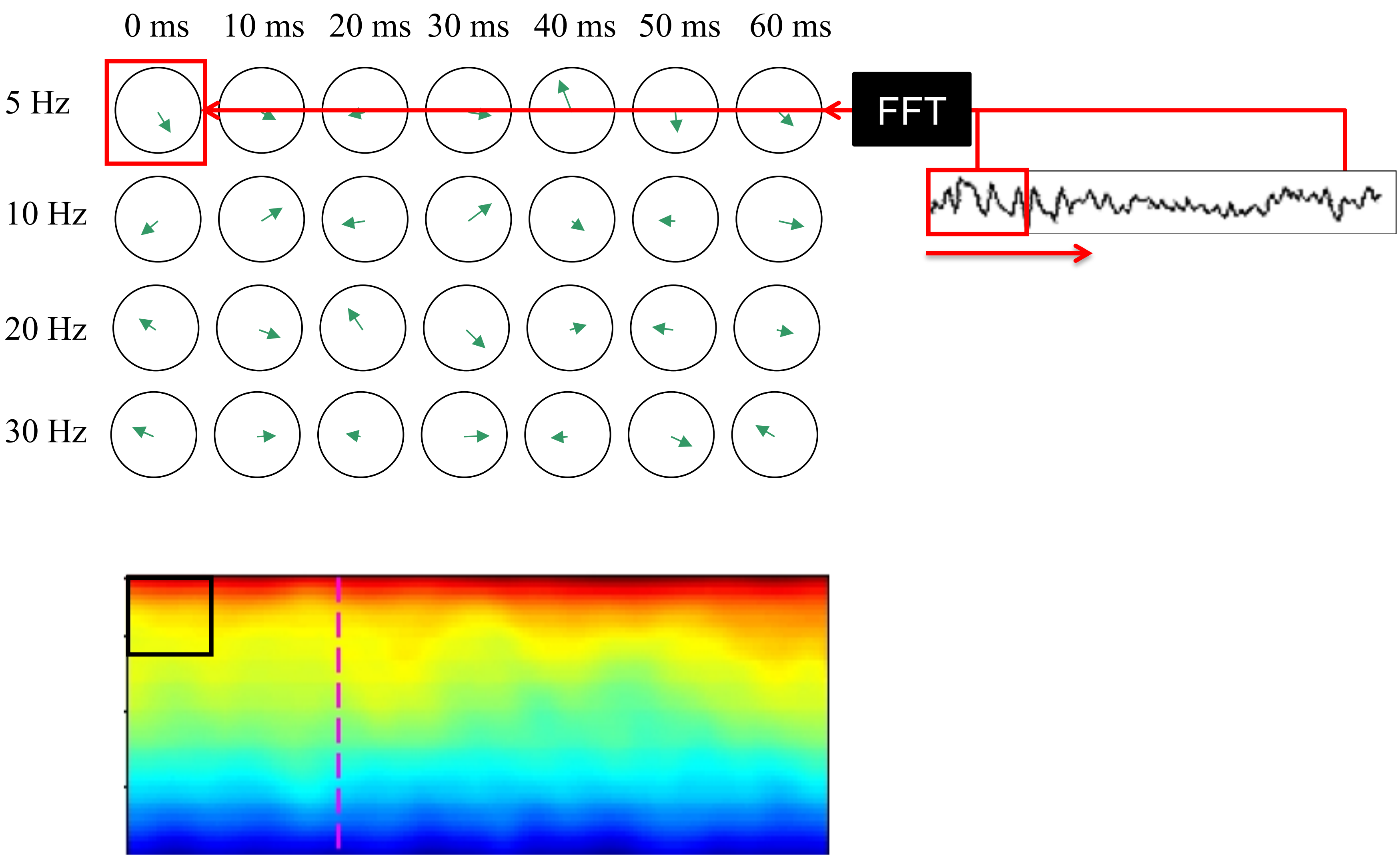



Evoked potential

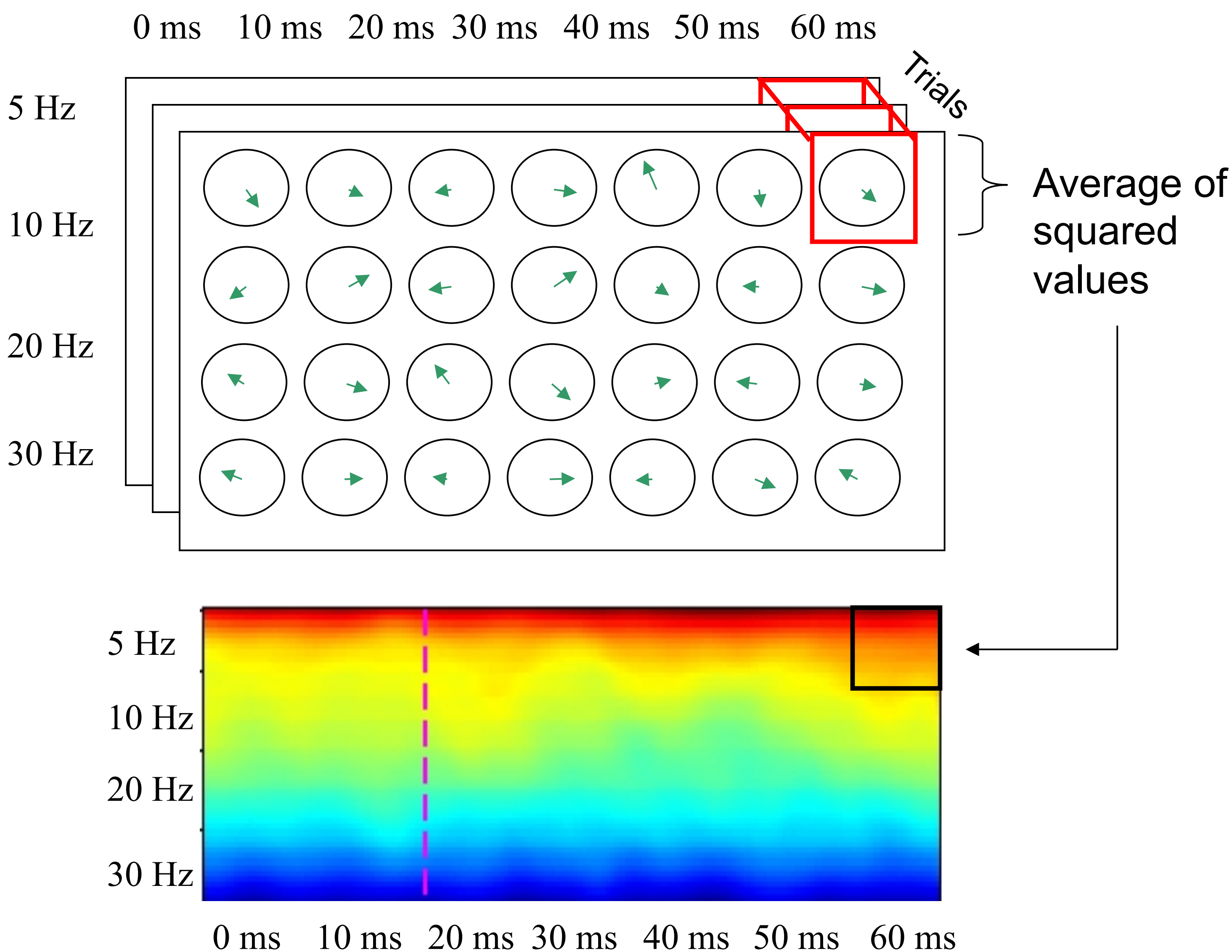



Chirp

# Spectrogram or ERSP



# Spectrogram or ERSP



# Power spectrum and event-related spectral (perturbation)

$$ERS(f,t) = \frac{1}{n}\sum_{k=1}^{n}\left|S_k(f,t)\right|^2$$

Ensemble average

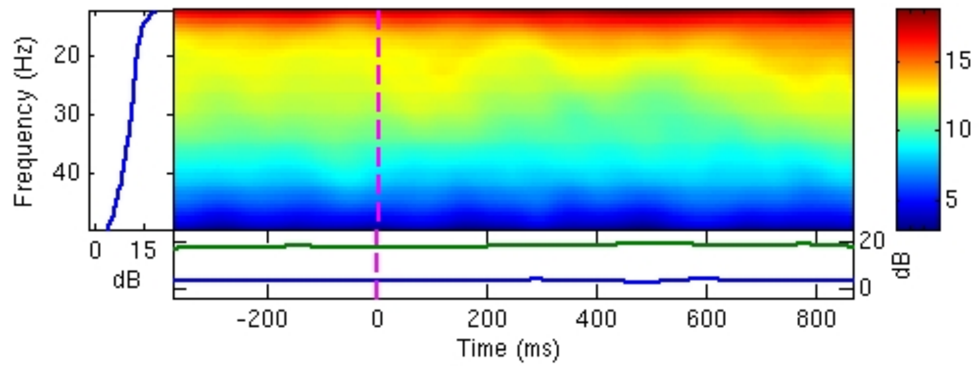
Complex number

Scaled to dB $10\mathrm{Log}_{10}$

Here, there are *n* trials
Each trial is time-locked to the same *event* (hence “event-related” spectrum)
The ERS is the average power across event-locked trials

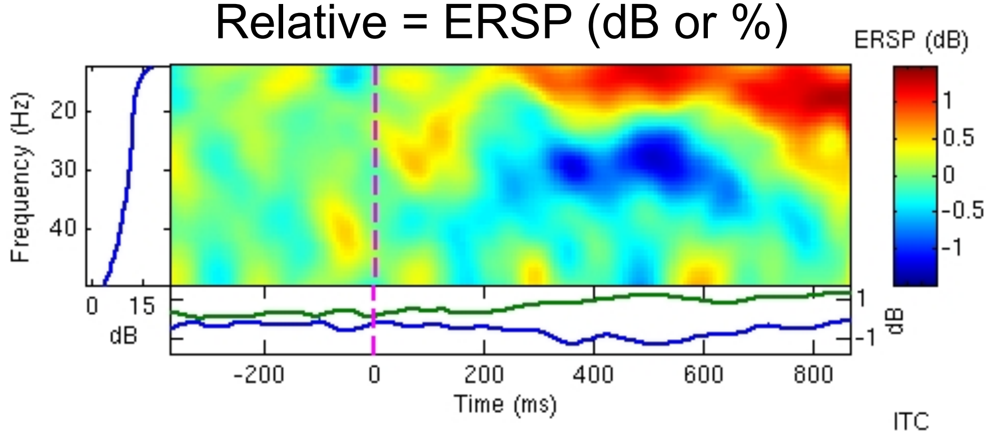
# Absolute versus relative power

Absolute = ERS




To compute the ERSP, we just subtract the pre-stimulus ERS from the whole trial

Relative = ERSP (dB or %)



# The Uncertainty Principle

A signal <u>cannot</u> be localized arbitrarily well <u>both</u> in time/position <u>and</u> in frequency/momentum.

There exists a lower bound to the <u>*Heisenberg product*</u>:

$$\Delta t \, \Delta f \geq 1/(4\pi)$$

or $$\Delta f \geq 1/(4\pi \Delta t)$$

Werner Karl Heisenberg
(1901 – 1976)

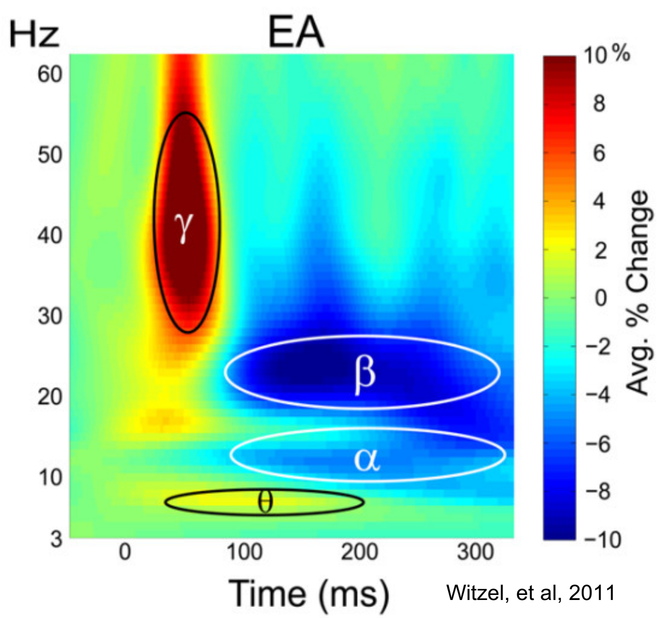
e.g. here are two possible ($\Delta f$, $\Delta t$) pairs:

*$\Delta f$ = 1Hz, $\Delta t$ = 80 msec or*
*$\Delta f$ = 2Hz, $\Delta t$ = 40 msec*

Note: $\Delta f$ means "difference between successive frequencies" or the inverse of the frequency resolution. Ditto for $\Delta t$.

# Time-Frequency Tradeoff

Natural biophysical processes may exhibit sustained changes in narrowband low-frequency oscillations along with rapidly-changing (e.g. “burst”) high-frequency oscillations.



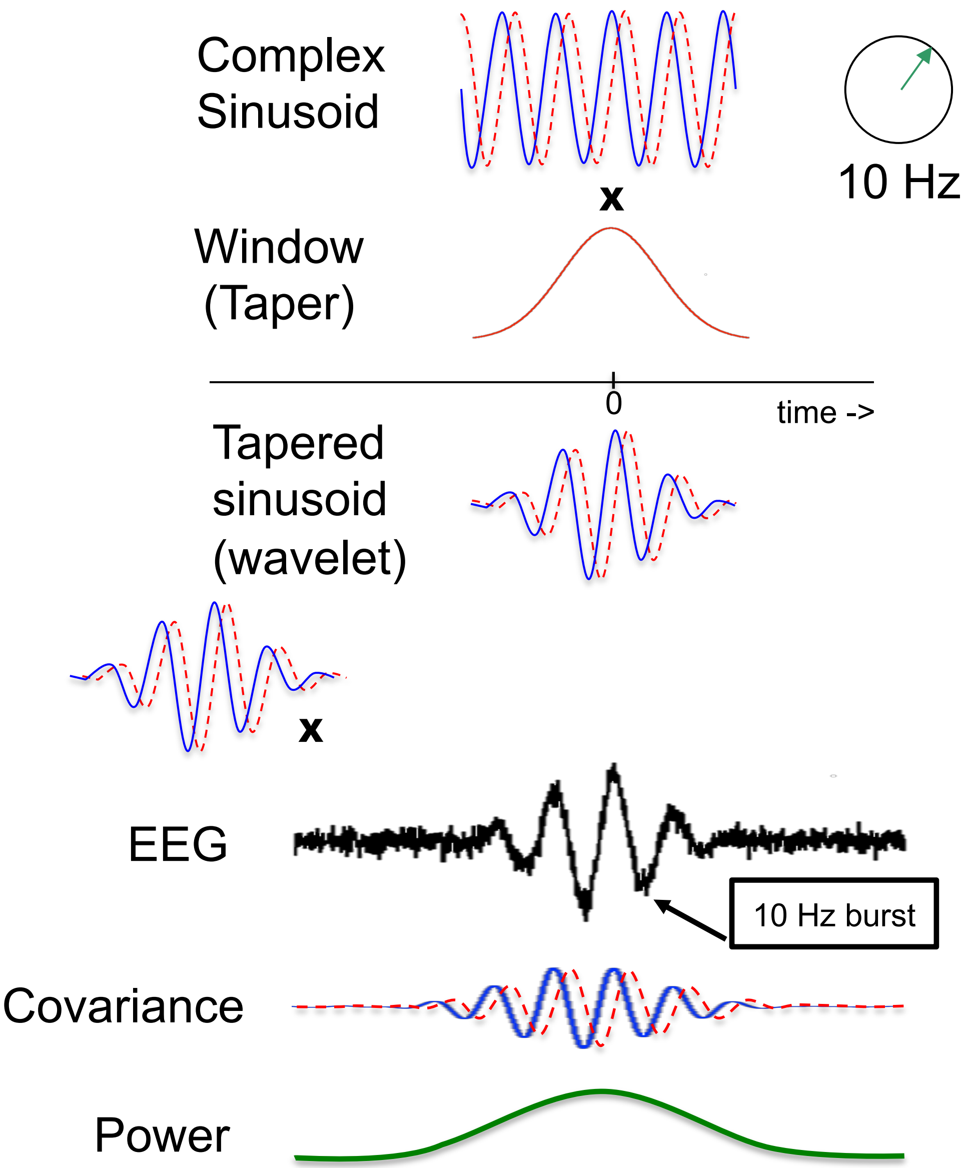

Witzel, et al, 2011

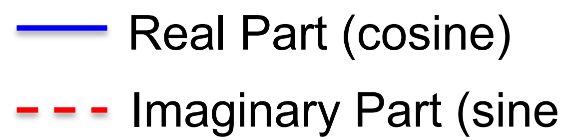
The Short-Time Fourier Transform has a constant temporal resolution for all frequencies.

Can we adapt the time-frequency resolution tradeoff for individual frequencies to improve spectral estimation?

**Yes, we can!**

# Wavelet Analysis

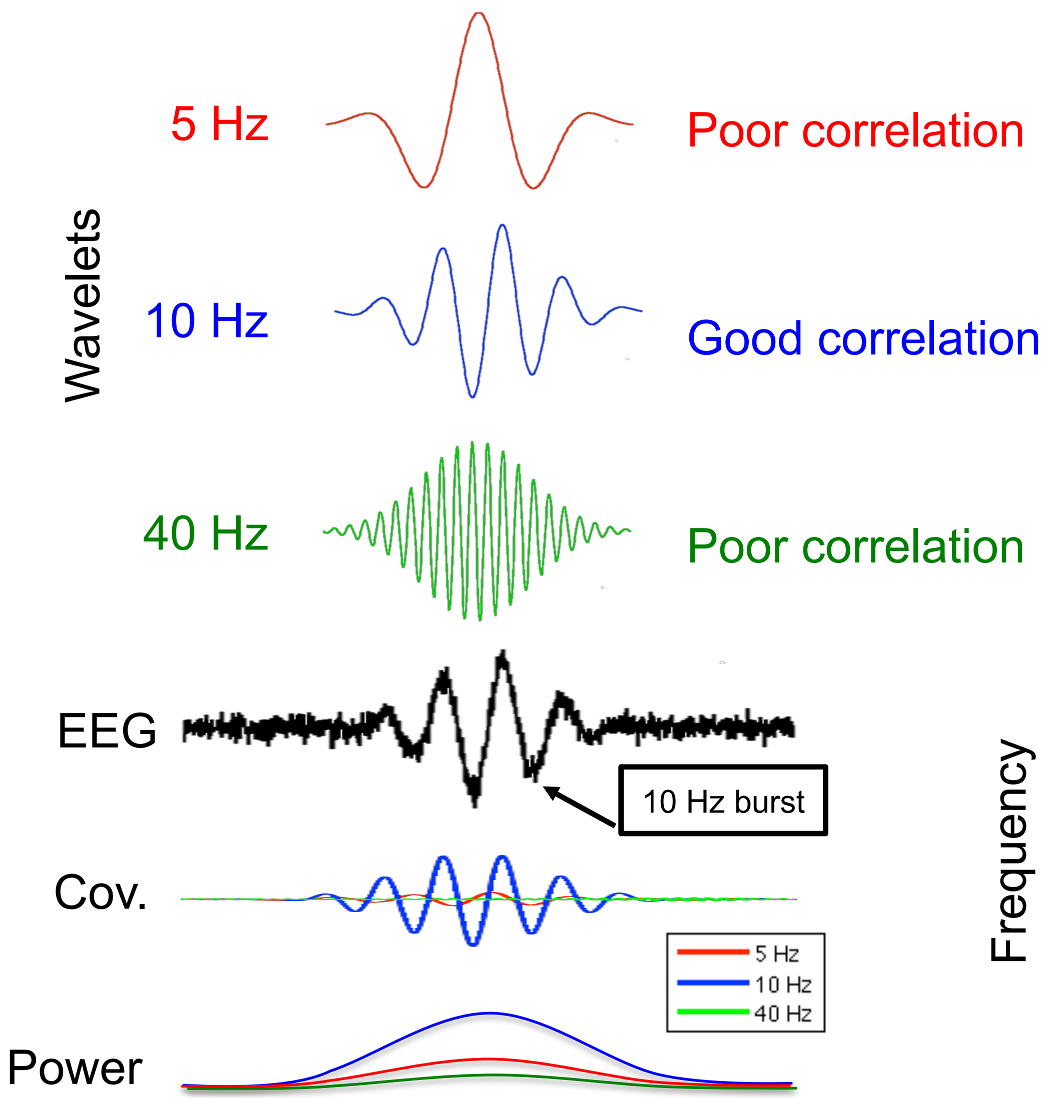


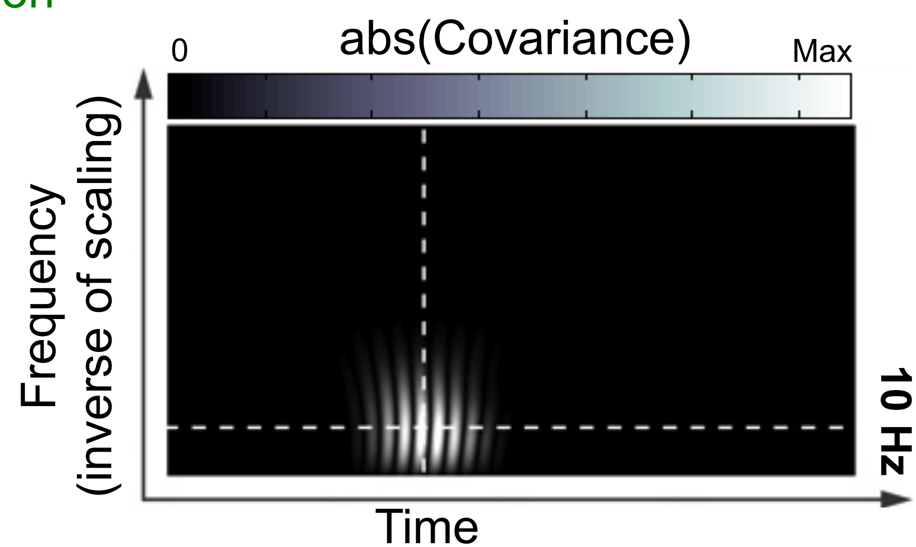

We estimate the time-varying power at 10 Hz by convolving EEG signal with a tapered 10 Hz complex sinusoid (Morlet wavelet)

$|S(f,t)| = |Re + Im|$

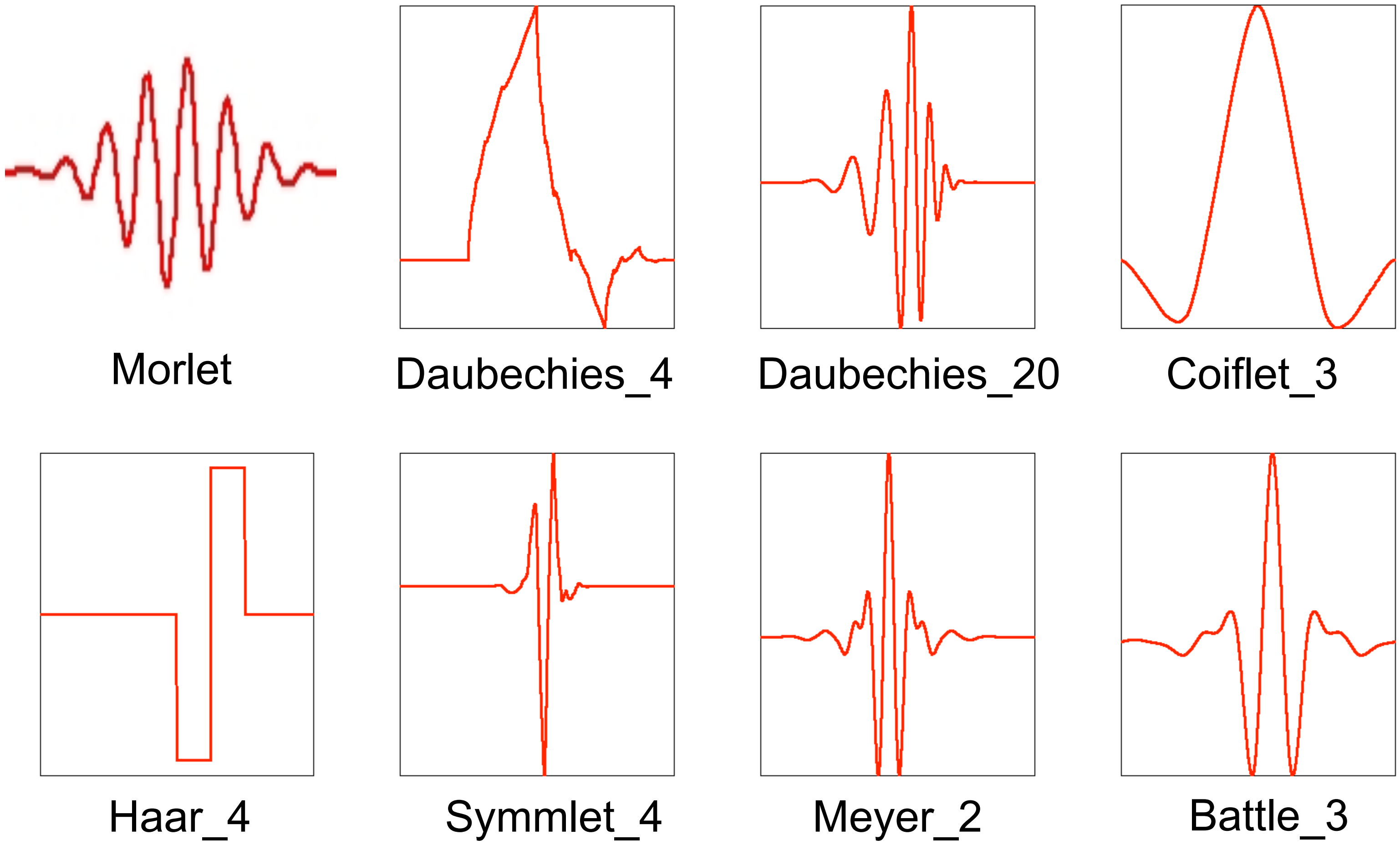
# Wavelet Time-Frequency Image




By convolving stretched and scaled versions of the “mother” wavelet with the EEG signal, we determine the time-frequency distribution of power

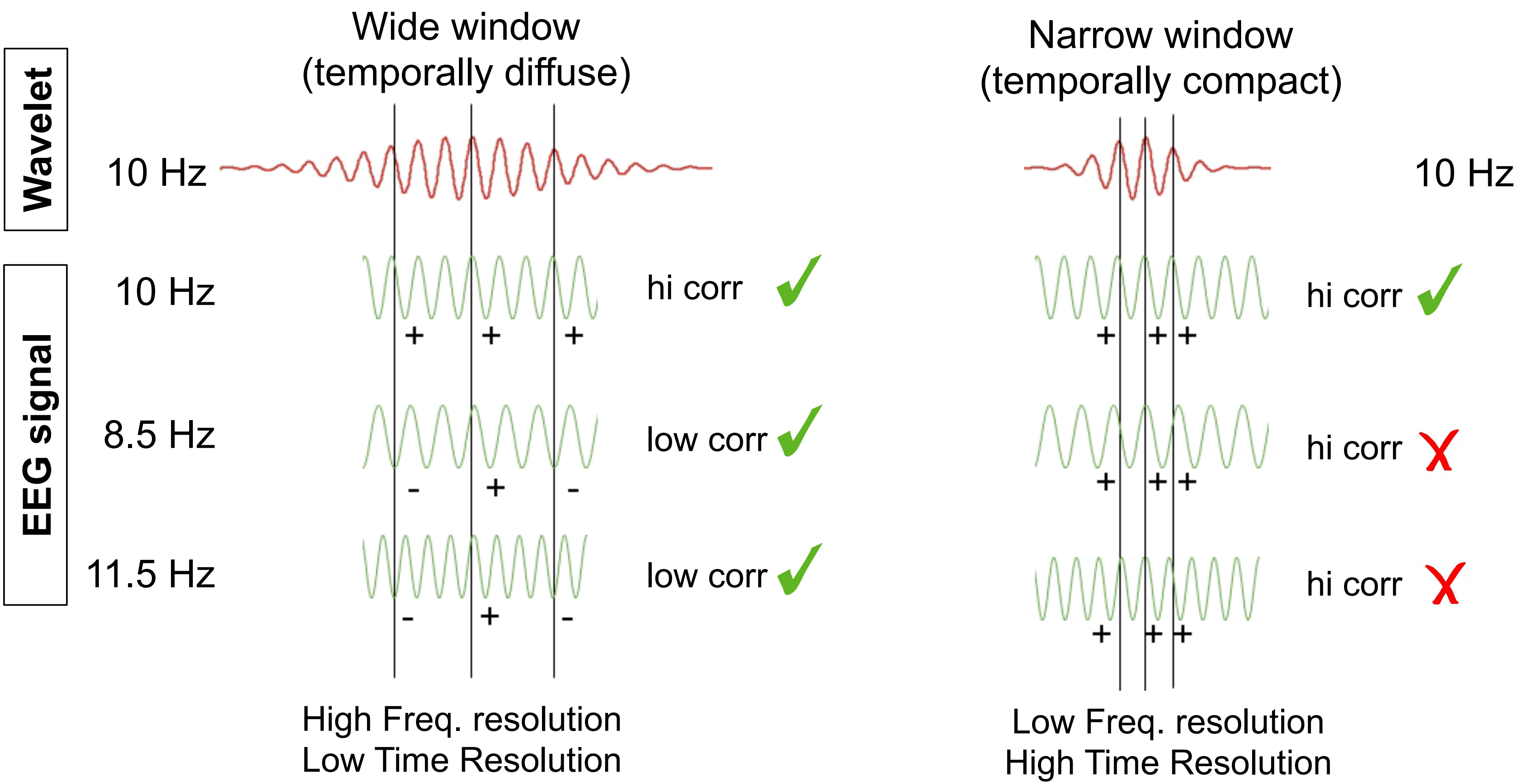


# Some Wavelet Families



# Trading Frequency for Time

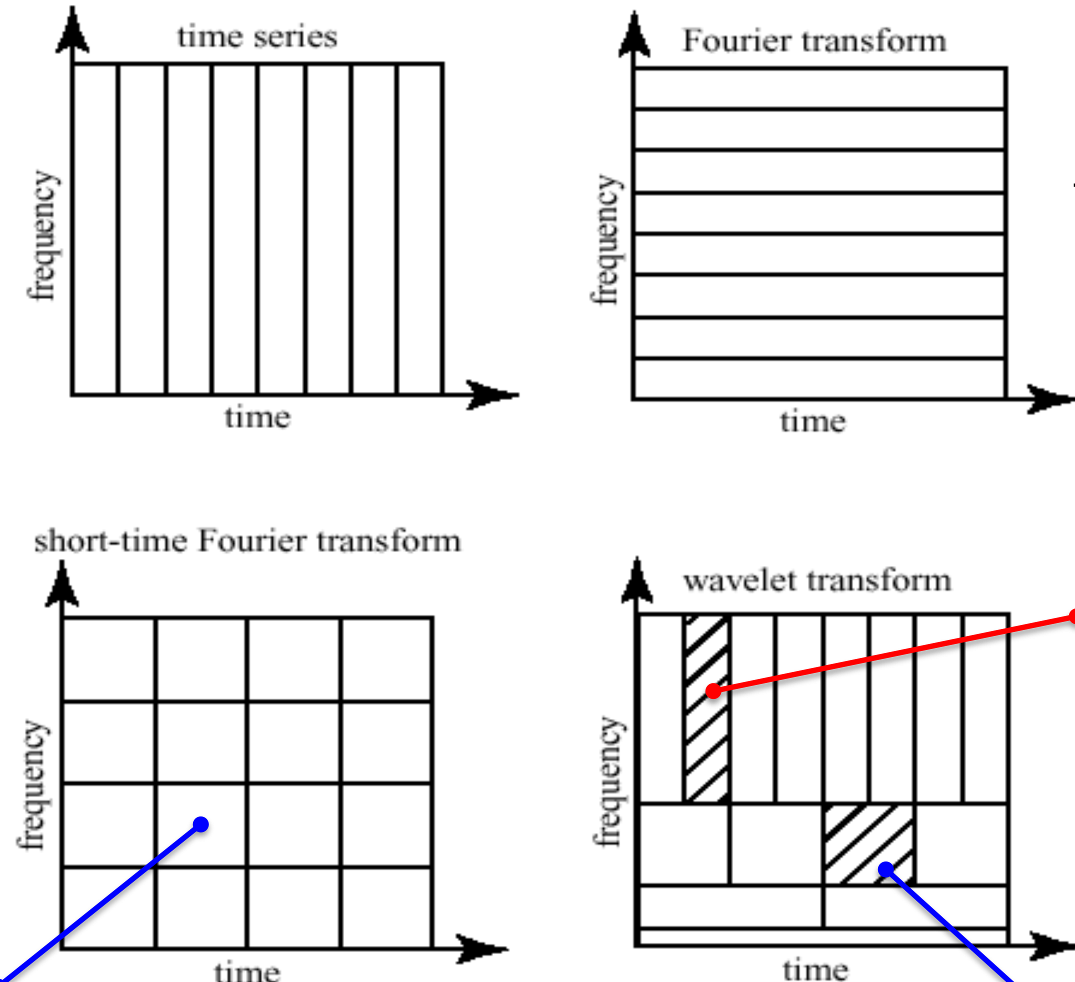
(and vice versa)



# FFT versus Wavelets

fixed time res
no freq res

fixed freq res
no time res




high time res
low freq res

fixed time res
fixed freq res

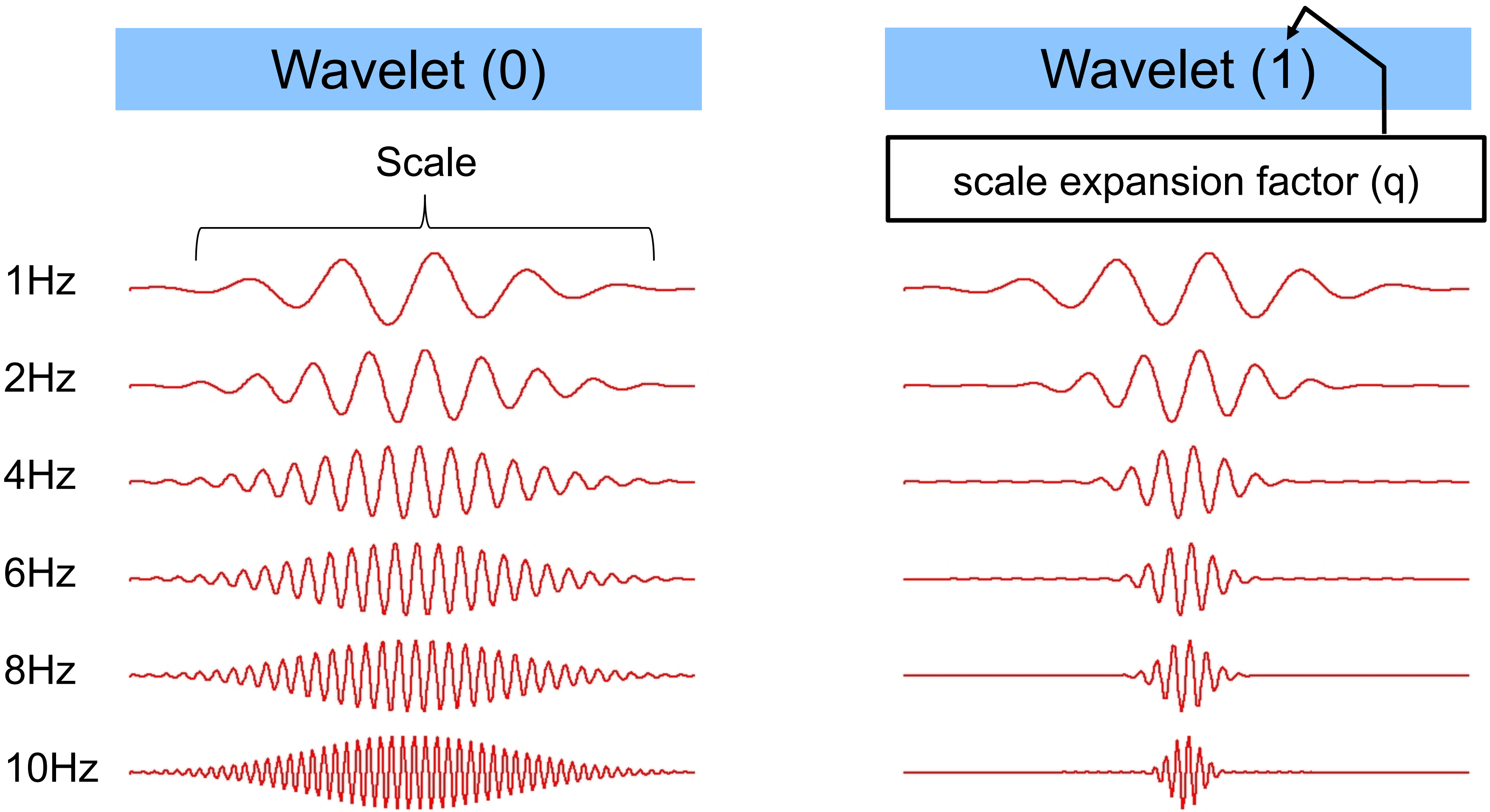
variable time res
variable freq res

equal time
and freq
resolution

equal time
and freq
resolution

Adapted from http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf, p.10

# Wavelet scale expansion factor

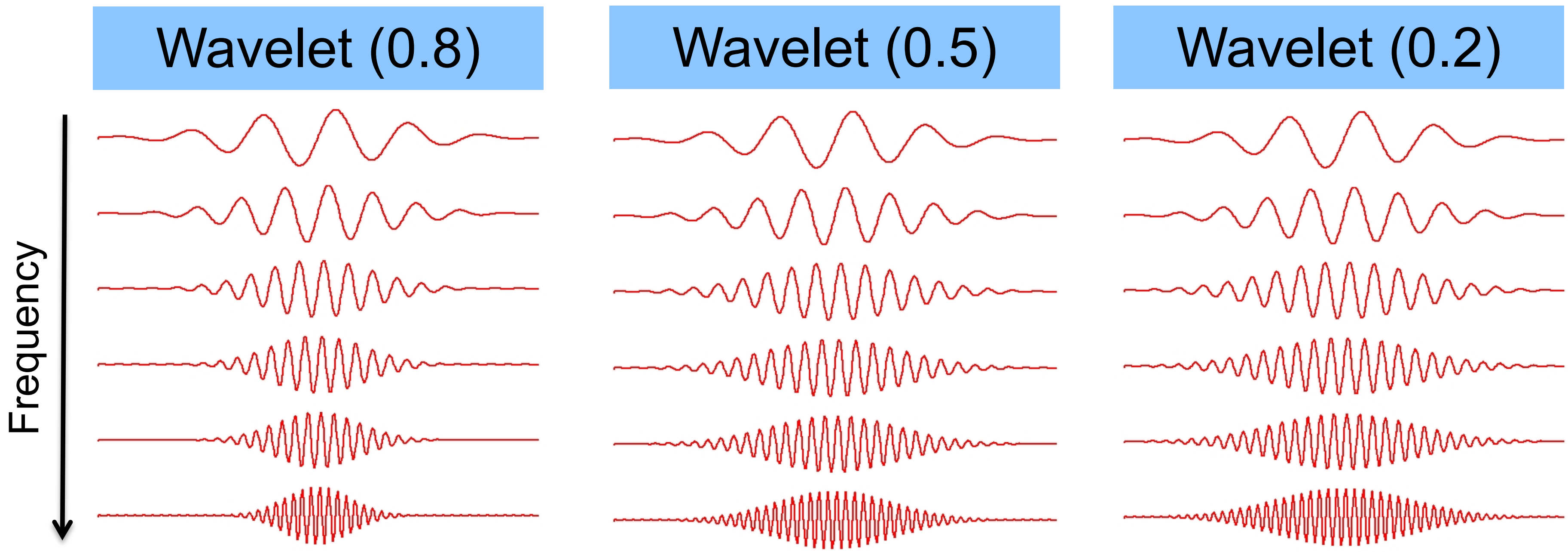



constant window size (time resolution) for increasing frequency → increasing # cycles with frequency.

window size decreases by a factor of 2 for each octave (power of 2) → constant # of cycles at each frequency

# Wavelet scale expansion factor

Larger expansion factor produces larger scale decrements (increased time resolution, lower frequency resolution) for increasing frequency

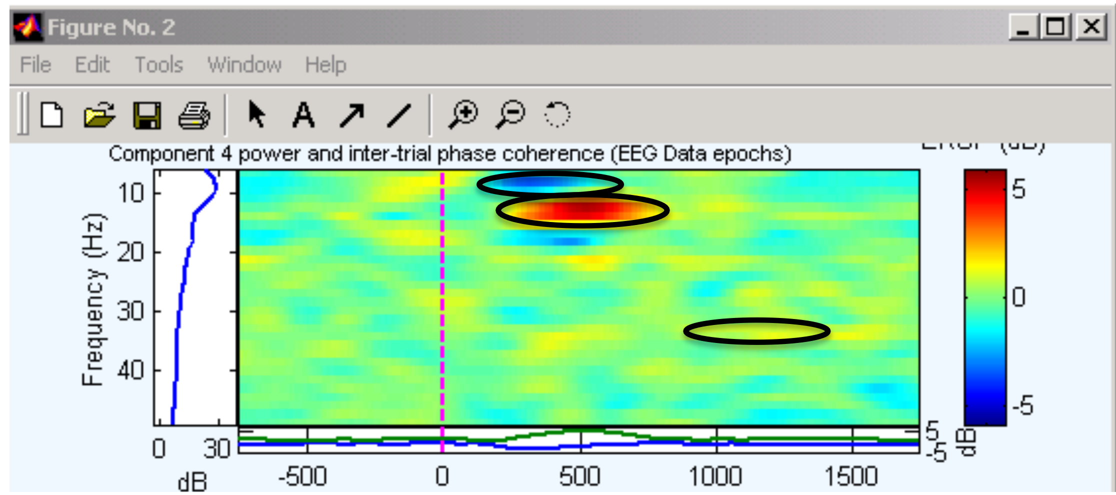



Number of cycles at highest frequency for an expansion factor of *q:*

$$C_{\mathrm{fmax}} = \frac{f_{max}}{f_{min}} C_{\mathrm{fmin}} (1-q)$$

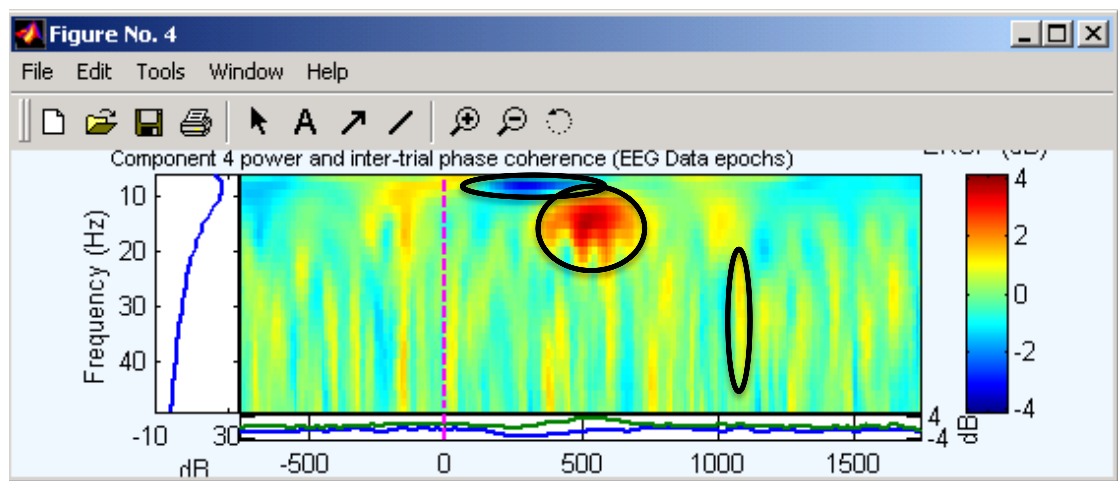
# Wavelet scale expansion factor

Wavelet(0)
(STFT)

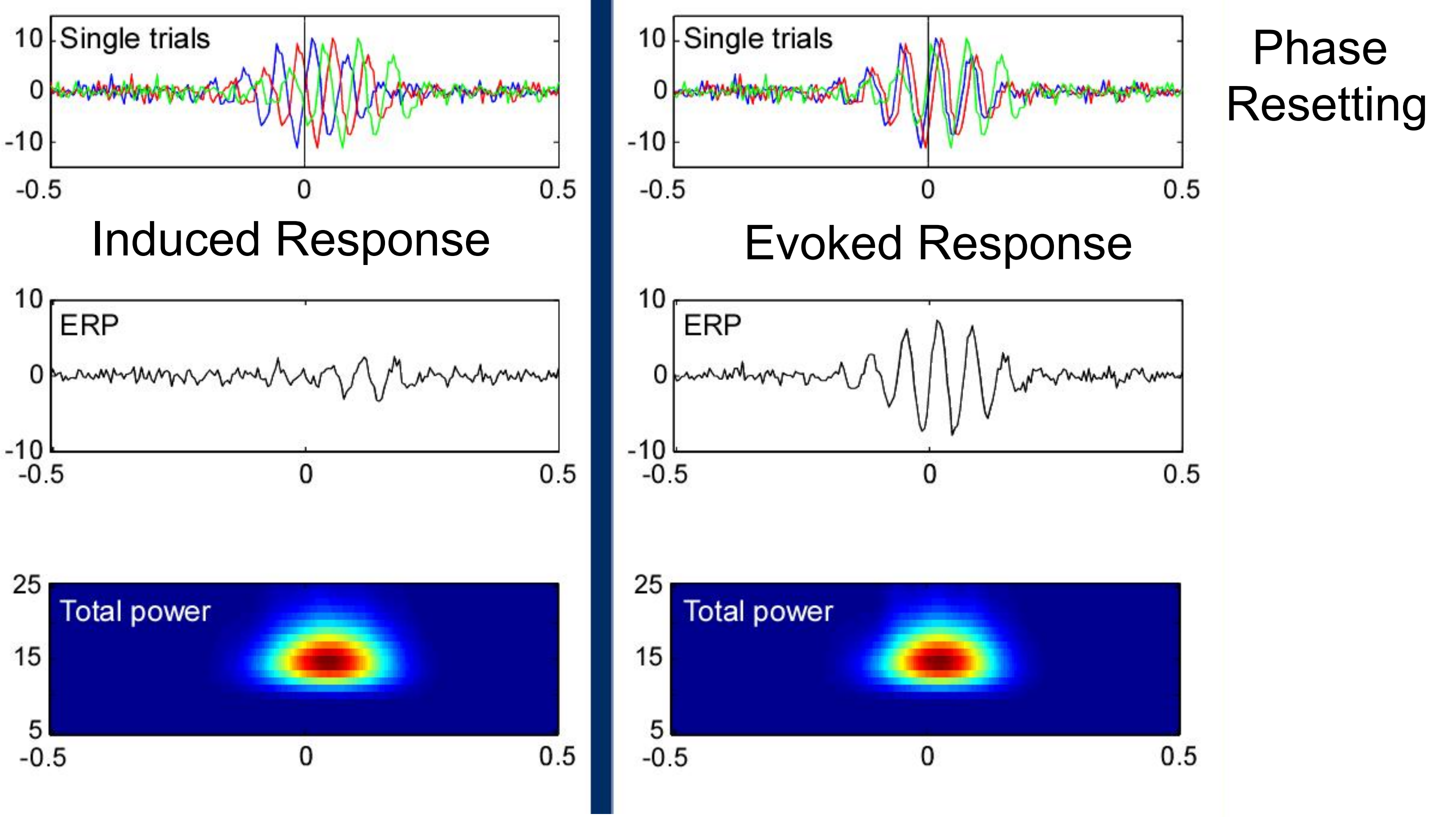



Wavelet (1)

scale expansion factor (q)



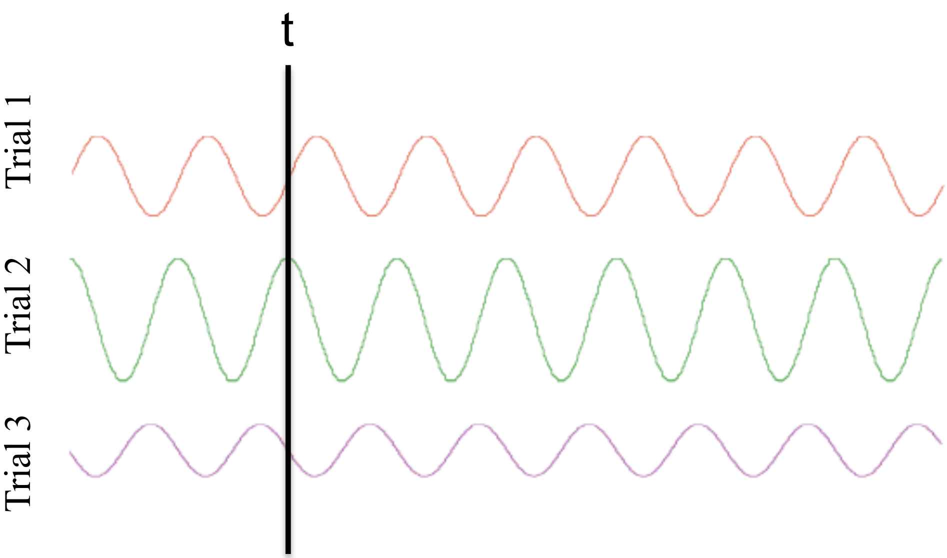
## Intertrial Coherence (ITC)

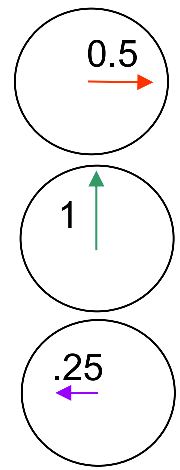



Slide courtesy of Stefan Debener

# Inter-Trial Coherence (ITC)

Tallon-Baudry, et al, 1996



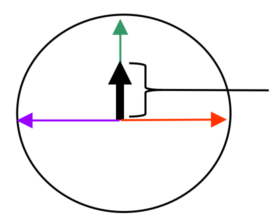




| | |
|---|---|
| amplitude 0.5 | phase 0 |
| amplitude 1 | phase 90 |
| amplitude 0.25 | phase 180 |

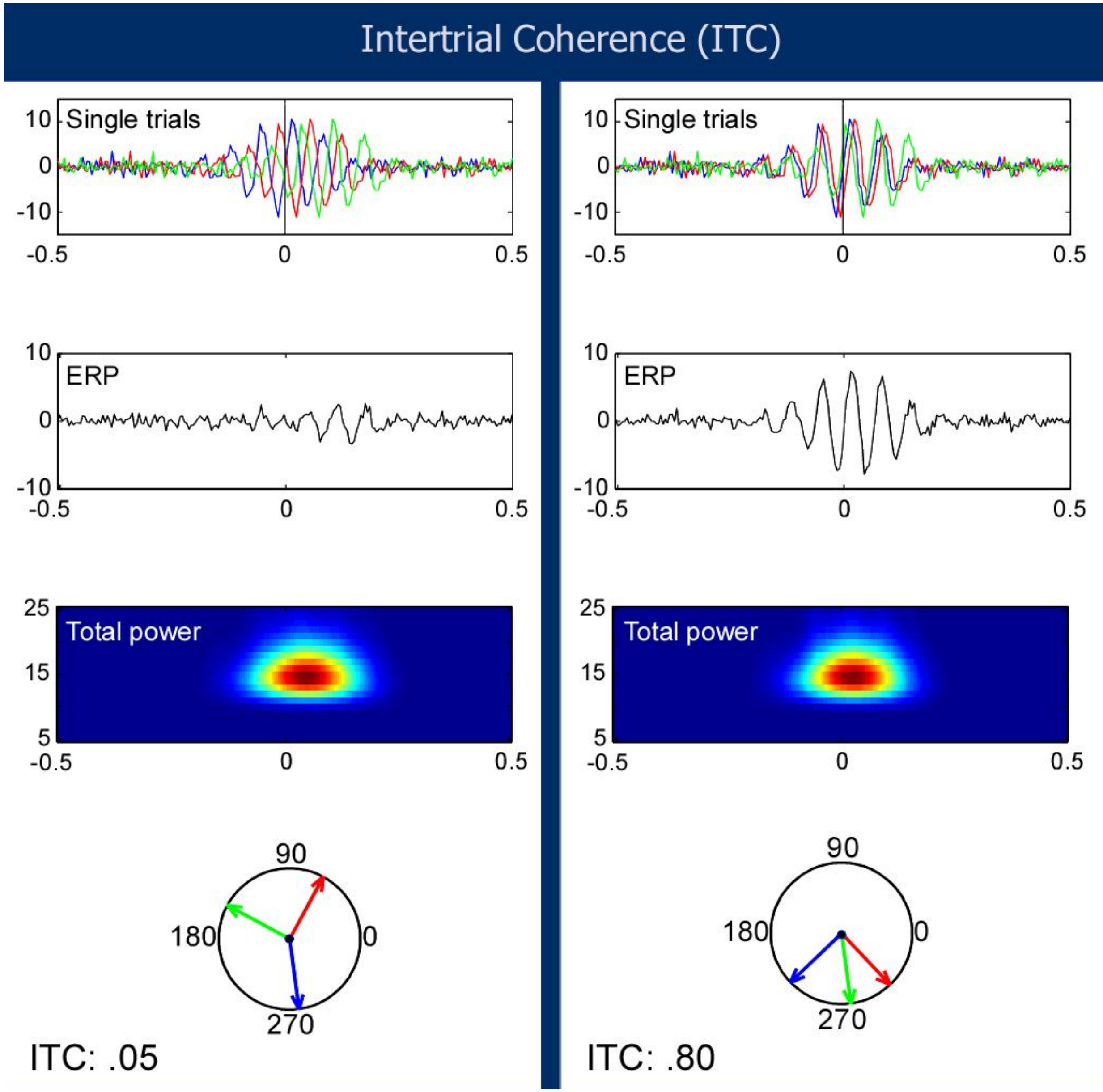
POWER = mean(amplitudes$^2$ )

0.44 or –8.3 dB

ITC = |mean(norm'd phase vectors)|




```
>> EEG = pop_newtimef(EEG,..., 'plotitc','on');
```



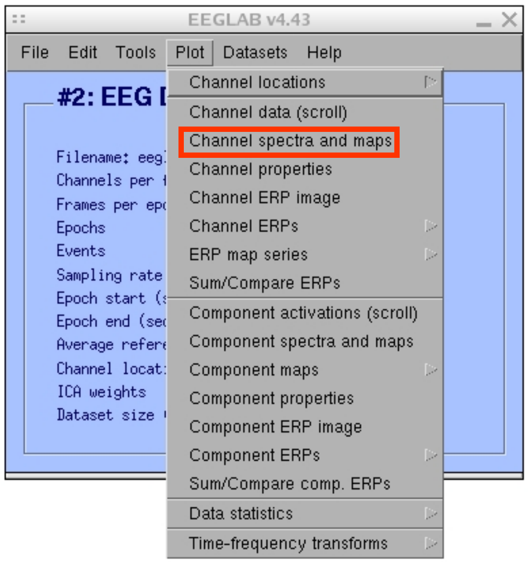

Phase Resetting

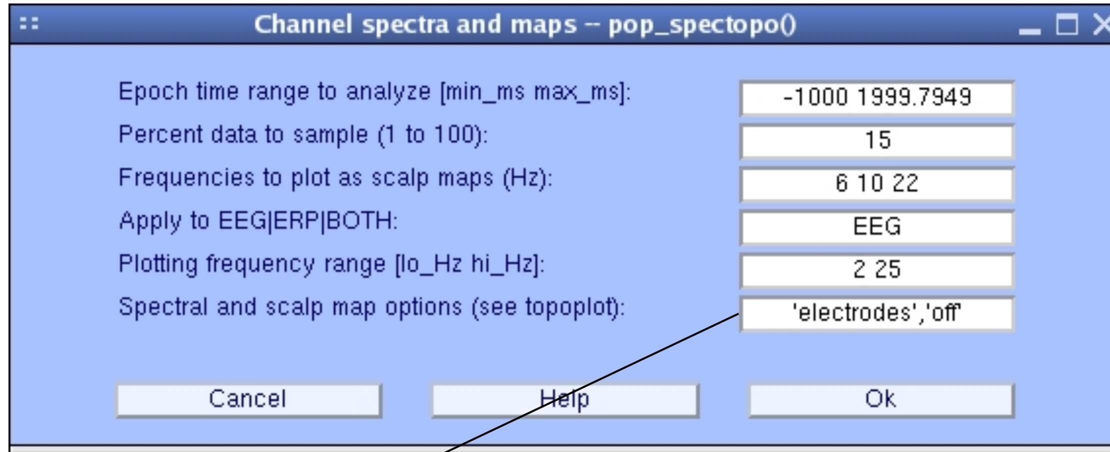
Slide courtesy of Stefan Debener

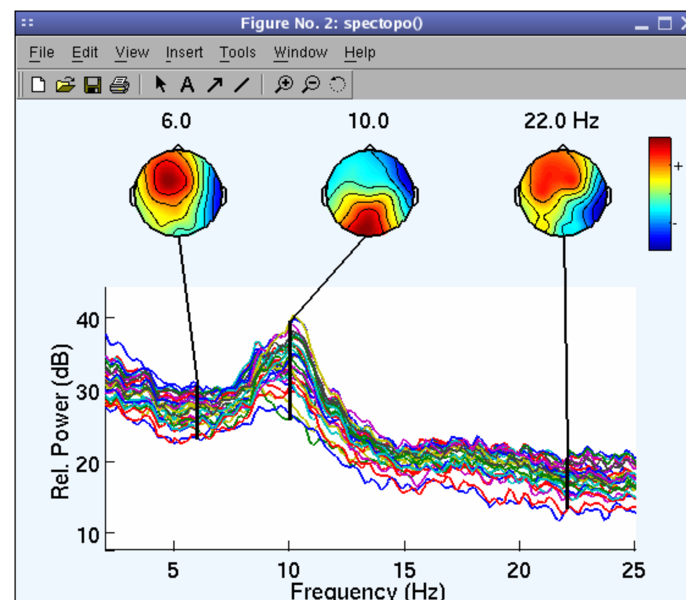
# Time-Frequency Analysis of Biophysical Time series:

# Practicum

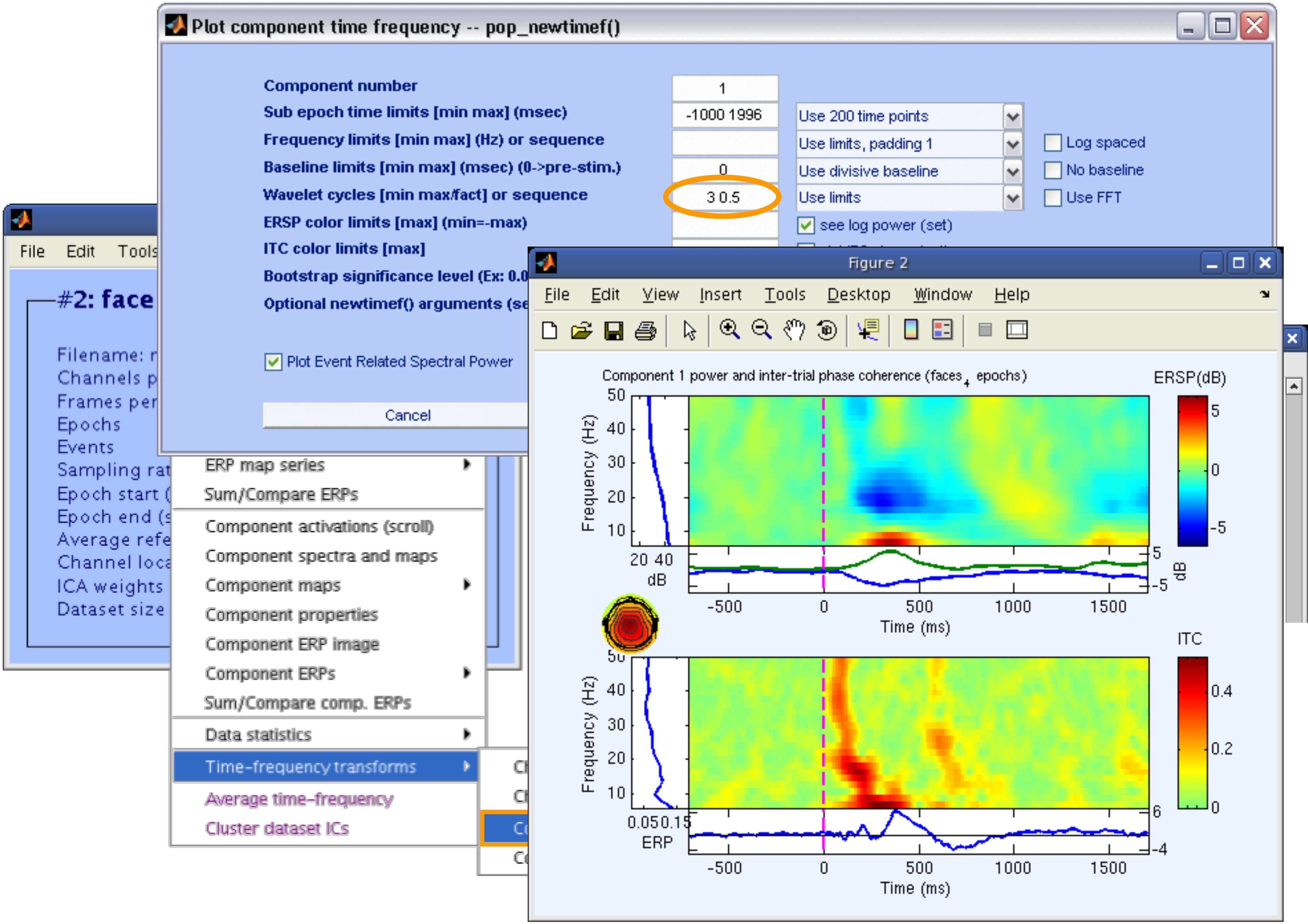
# Plot periodogram (spectrum) using Welch's method



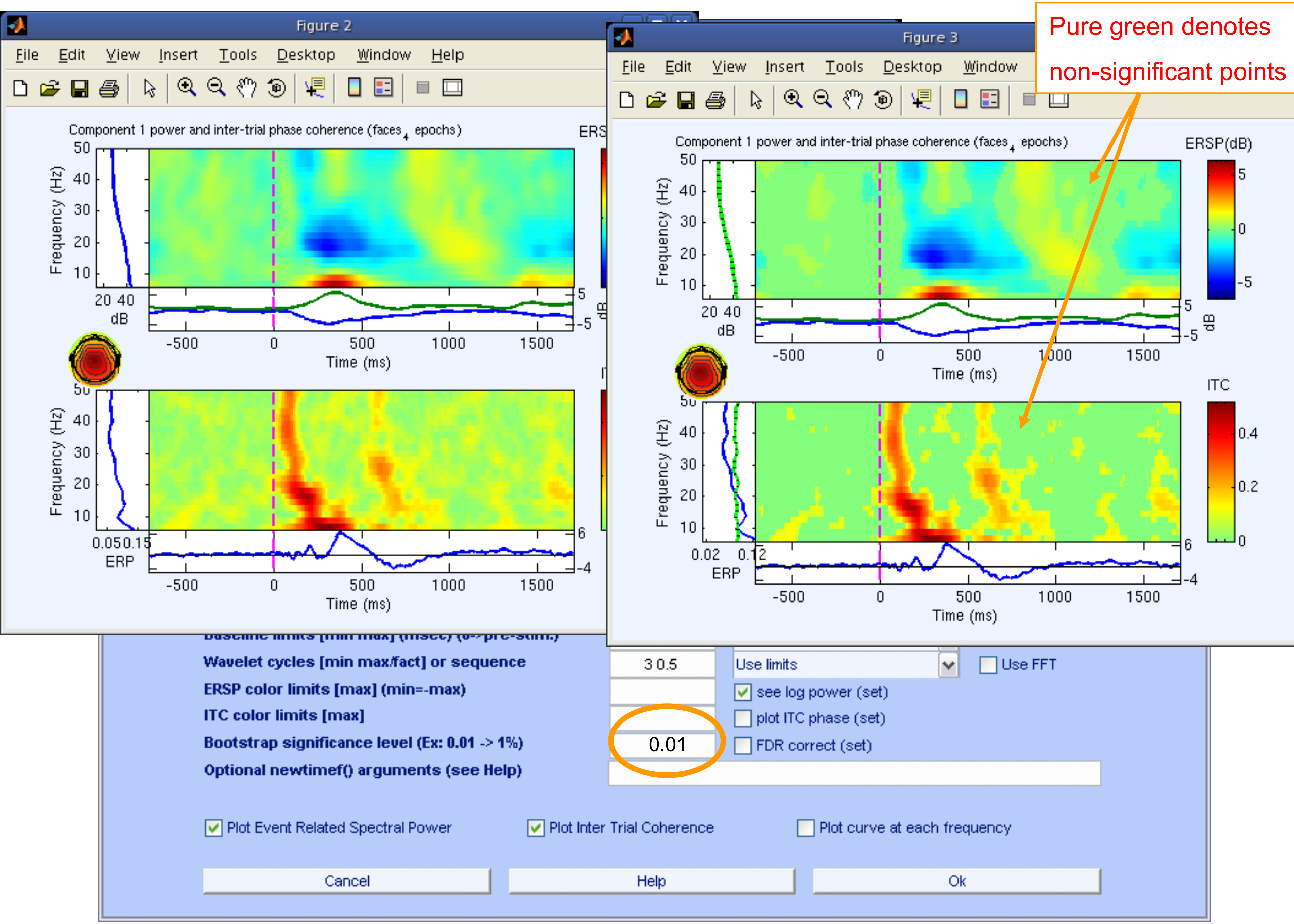




**'winsize' , 256** (change FFT window length)
**'nfft' , 256** (change FFT padding)
**'overlap' , 128** (change window overlap)



Plot component time frequency -- pop_newtimef()
Component number
1
Sub epoch time limits [min max] (msec)
-1000 1996
Use 200 time points
Frequency limits [min max] (Hz) or sequence
Use limits, padding 1
Log spaced
Baseline limits [min max] (msec) (0->pre-stim.)
0
Use divisive baseline
No baseline
Wavelet cycles [min max/fact] or sequence
3 0.5
Use limits
Use FFT
ERSP color limits [max] (min=-max)
see log power (set)
ITC color limits [max]
Optional newtimef() arguments
Plot Event Related Spectral Power
Cancel
File Edit Tools
#2: face
Filename:
Channels
Frames per
Epochs
Events
Sampling rat
Epoch start
Epoch end
Average refe
Channel loca
ICA weights
Dataset size
ERP map series
Sum/Compare ERPs
Component activations (scroll)
Component spectra and maps
Component maps
Component properties
Component ERP image
Component ERPs
Sum/Compare comp. ERPs
Data statistics
Time-frequency transforms
Average time-frequency
Cluster dataset ICs
Figure 2
File Edit View Insert Tools Desktop Window Help
Component 1 power and inter-trial phase coherence (faces, epochs)
ERSP(dB)
Frequency (Hz)
Time (ms)
dB
ITC
ERP


Pure green denotes
non-significant points
Figure 2
Figure 3
File Edit View Insert Tools Desktop Window Help
Component 1 power and inter-trial phase coherence (faces epochs)
ERSP(dB)
ITC
Frequency (Hz)
Time (ms)
dB
ERP
Wavelet cycles [min max/fact] or sequence
ERSP color limits [max] (min=-max)
ITC color limits [max]
Bootstrap significance level (Ex: 0.01 -> 1%)
Optional newtimef() arguments (see Help)
3 0.5
0.01
Use limits
Use FFT
see log power (set)
plot ITC phase (set)
FDR correct (set)
Plot Event Related Spectral Power
Plot Inter Trial Coherence
Plot curve at each frequency
Cancel
Help
Ok

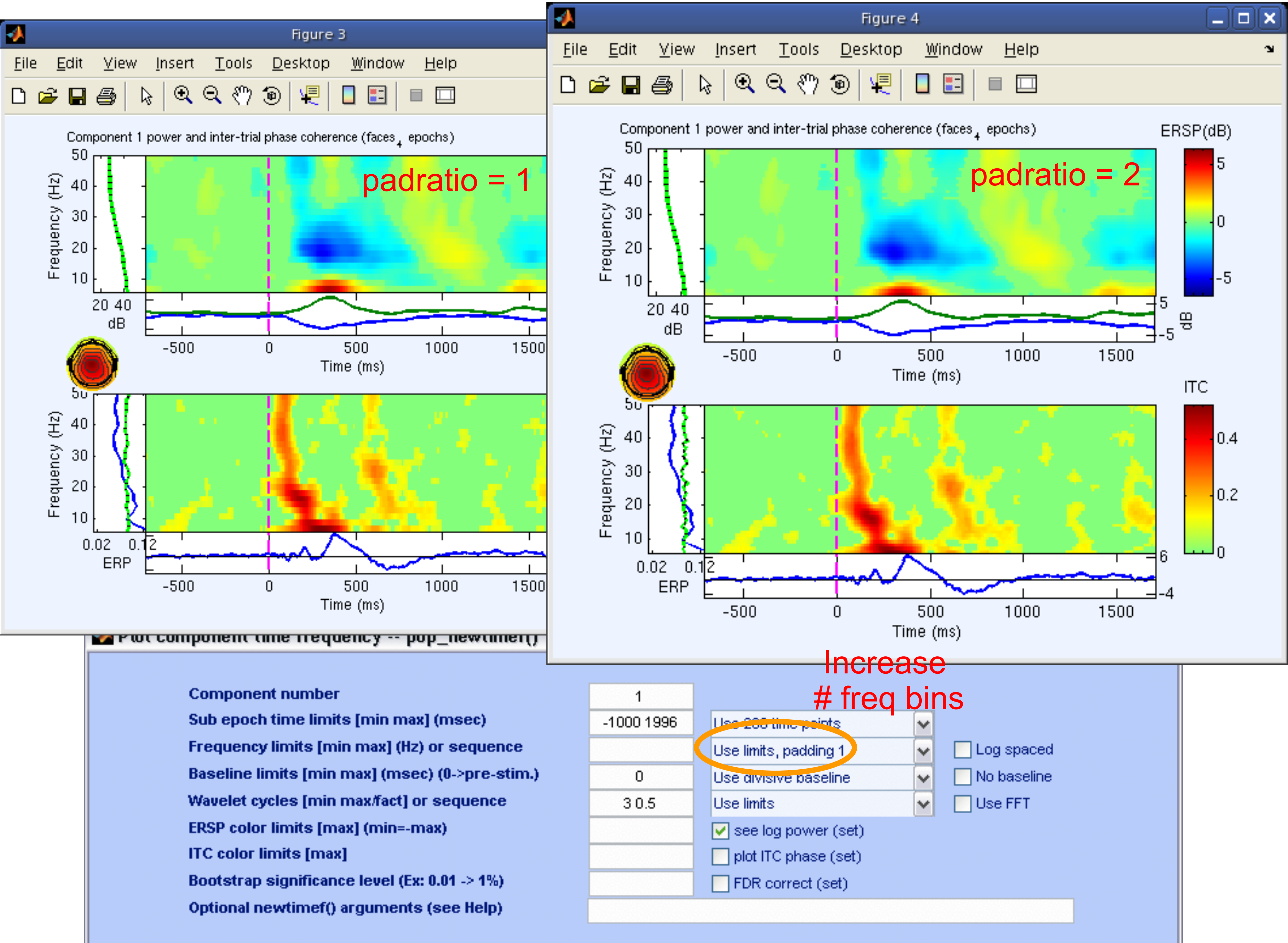
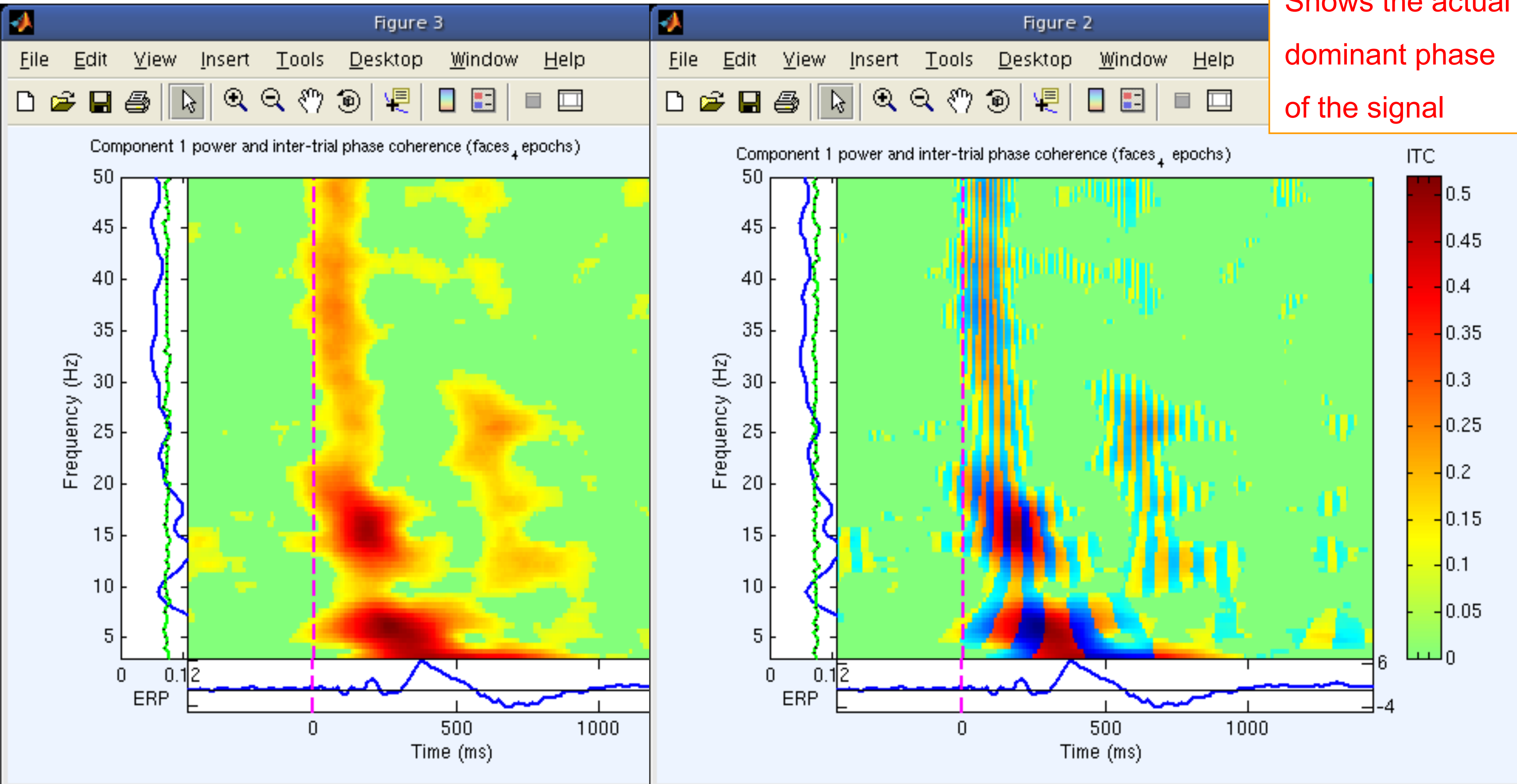
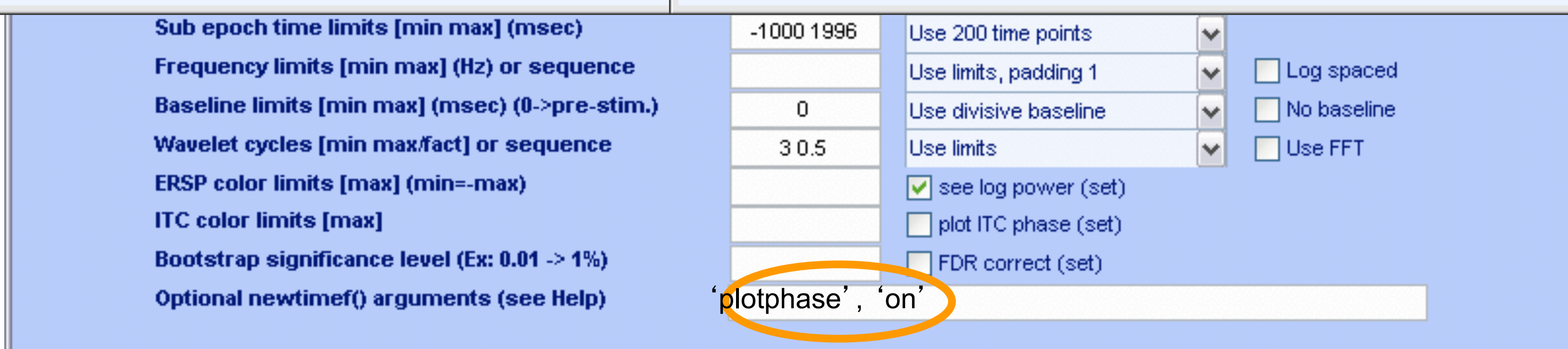


Figure 3
File Edit View Insert Tools Desktop Window Help
Component 1 power and inter-trial phase coherence (faces, epochs)
padratio = 1
Frequency (Hz)
dB
Time (ms)
ERP
Figure 4
File Edit View Insert Tools Desktop Window Help
Component 1 power and inter-trial phase coherence (faces, epochs)
padratio = 2
ERSP(dB)
ITC
Frequency (Hz)
dB
Time (ms)
ERP
Increase
# freq bins
Plot component time frequency -- pop_newtimef()
Component number
Sub epoch time limits [min max] (msec)
Frequency limits [min max] (Hz) or sequence
Baseline limits [min max] (msec) (0->pre-stim.)
Wavelet cycles [min max/fact] or sequence
ERSP color limits [max] (min=-max)
ITC color limits [max]
Bootstrap significance level (Ex: 0.01 -> 1%)
Optional newtimef() arguments (see Help)
1
-1000 1996
0
3 0.5
Use 200 time points
Use limits, padding 1
Use divisive baseline
Use limits
Log spaced
No baseline
Use FFT
see log power (set)
plot ITC phase (set)
FDR correct (set)

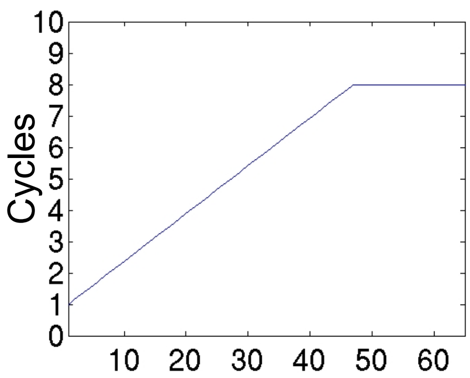
Shows the actual dominant phase of the signal

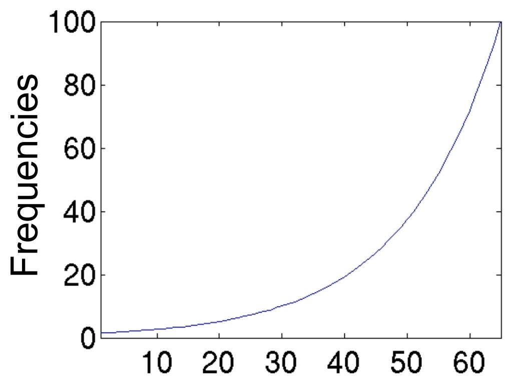


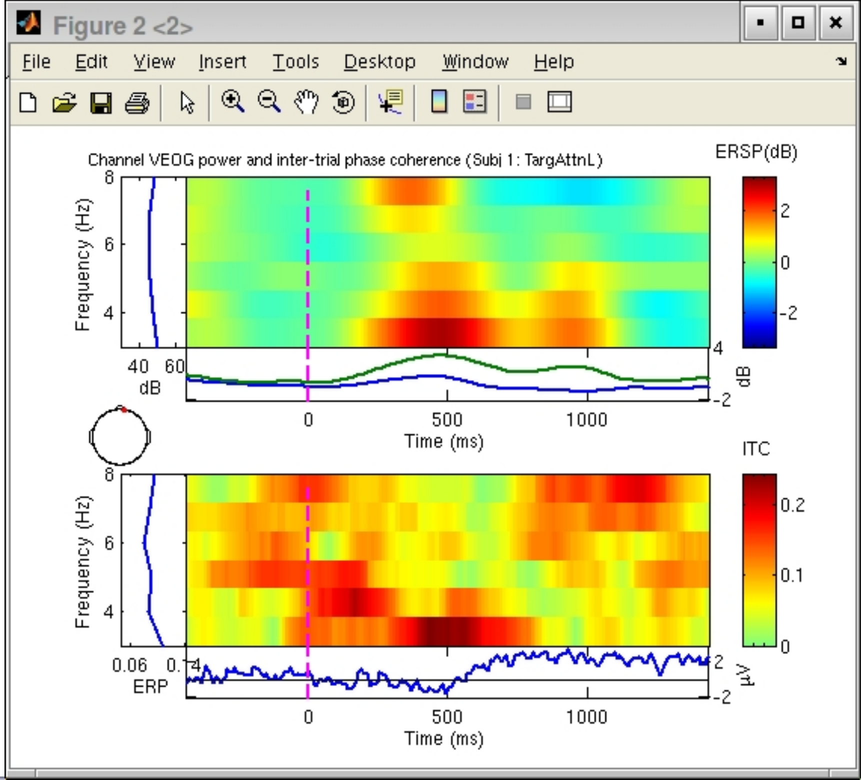



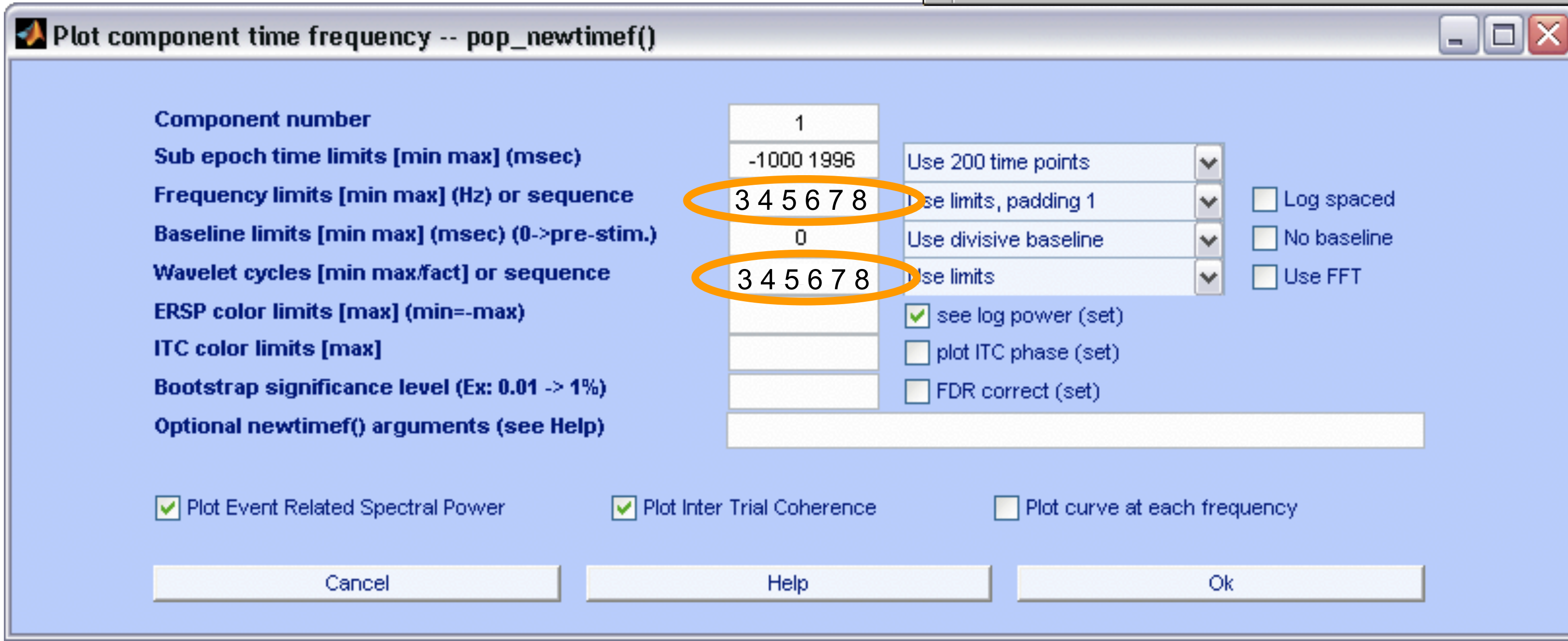
## To visualize both low and high frequencies

```
freqs  = exp(linspace(log(1.5), log(100), 65));
cycles = [ linspace(1, 8, 47) ones(1,18)*8 ];
```

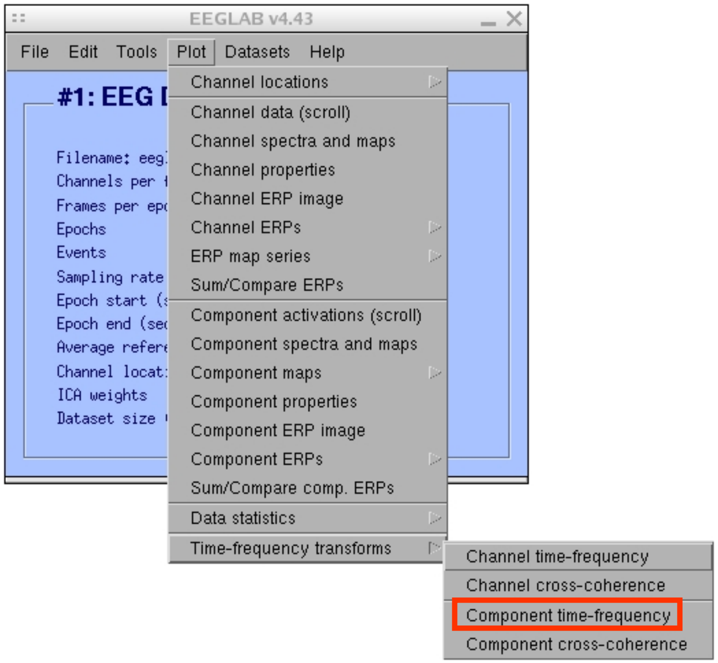


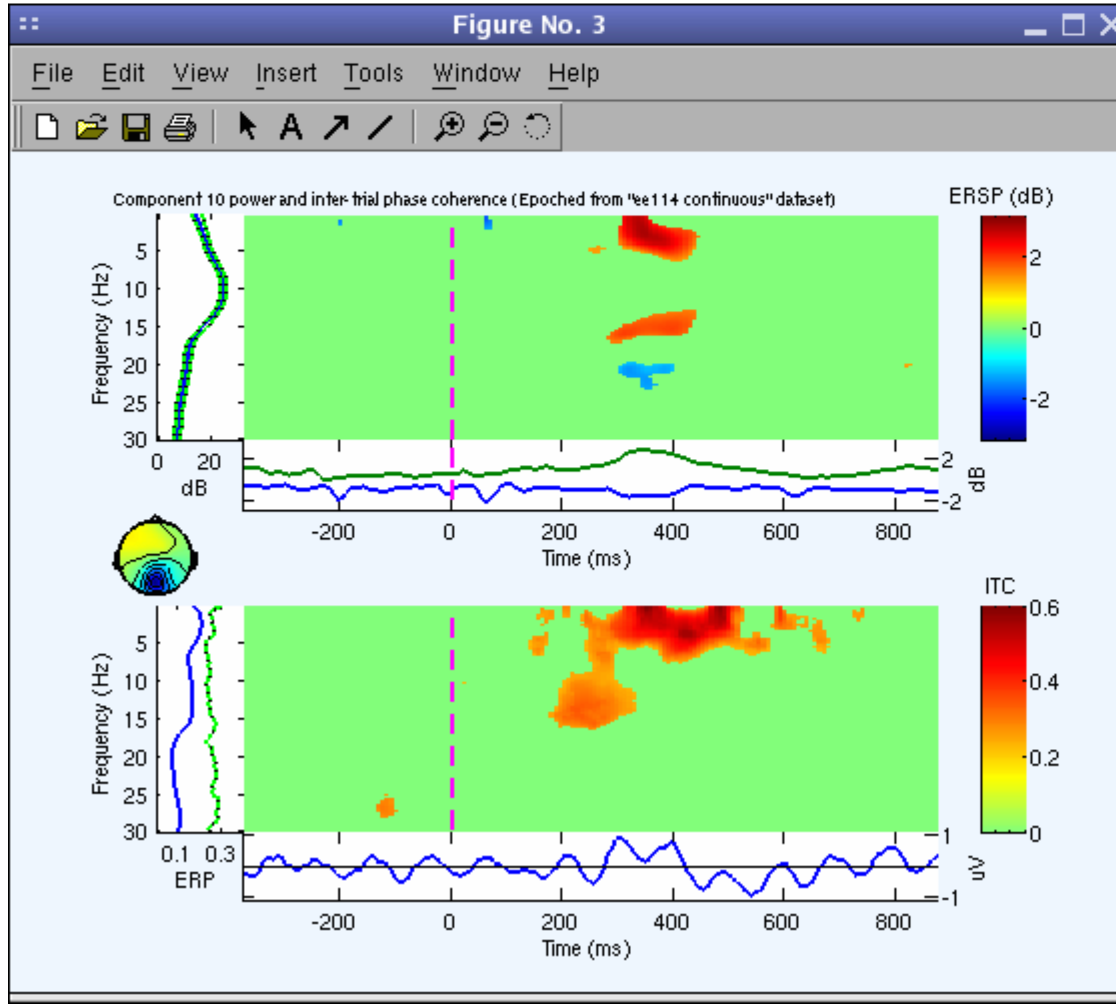



# Component time-frequency



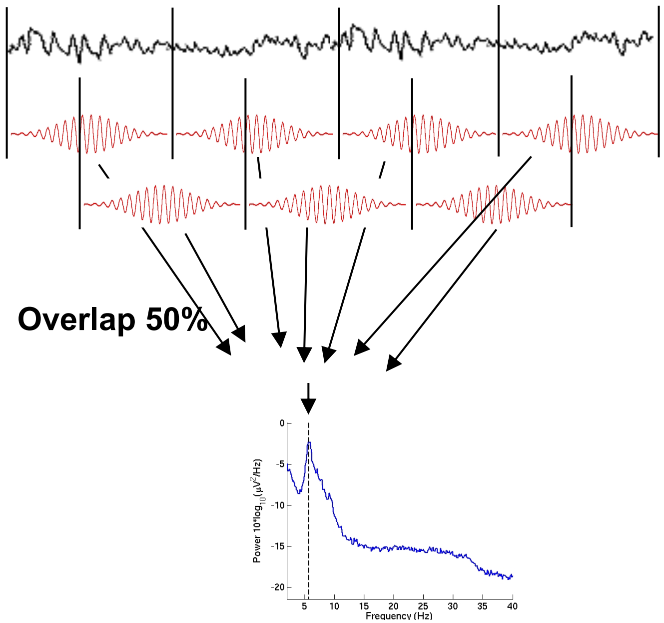
# Exercise

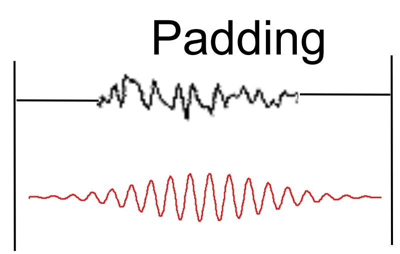
- **ALL**
  Start EEGLAB, from the menu:
  load
  <eeglab_root>/*sample_data/eeglab_data_epochs_ica.set*
  or your own data

- **Novice**
  From the GUI, Plot spectral decomposition with 100% data and 50% overlap ('overlap'). Try reducing window length ('winsize') and FFT length ('nfft')

- **Intermediate**
  Same as novice but using a command line call to the *pop_spectopo()* function. Use GUI then history to see a standard call ("eegh").

- **Advanced**
  Same as novice but using a command line call to the *spectopo()* function.



# Exercise - newtimef

- **Novice**
  From the GUI, pick an interesting IC and plot component ERSP. Try changing parameters window size, number of wavelet cycles, padratio,

- **Intermediate**
  From the command line, use newtimef() to tailor your time/frequency output to your liking. Look up the help to try not to remove the baseline, change baseline length and plot in log scale. Enter custom frequencies and cycles.

- **Advanced**
  Compare FFT, the different wavelet methods (see help), and multi-taper methods (use timef function not newtimef). Enter custom frequencies and cycles. Look up newtimef help to compare conditions. Visualize single-trial time-frequency power using erpimage.