The Dynamic Brain I: Modeling Neural Dynamics and Interactions from M/EEG

\[ y(t) = Hs(t) + e(t) \]

\[ s(t) = f(s(t^-), u(t^-)) + v(t) \]

Tim Mullen
Swartz Center for Computational Neuroscience
Institute for Neural Computation
Dept. of Cognitive Science
UC San Diego
# Outline

## Theoretical Foundations I

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**Practicum: Hands-On Walkthrough of SIFT**
## Preview Outline (Sunday)

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Practicum: Hands-On Simulation-based training
The Dynamic Brain

- A key goal: To model temporal changes in neural dynamics and information flow that index and predict task-relevant changes in cognitive state and behavior

- Open Challenges:
  - Non-invasive measures (source inference)
  - Robustness and Validity (constraints & statistics)
  - Scalability (multivariate)
  - Temporal Specificity / Non-stationarity / Single-trial (dynamics)
  - Multi-subject Inference
  - Usability and Data Visualization (software)

Mullen, 2011
Modeling Brain Connectivity

- Model-based approaches mitigate the ‘curse of dimensionality’ by making some assumptions about the structure, dynamics, or statistics of the system under observation.

Box and Draper (1987):

“Essentially, all models are wrong, but some are useful [...] the practical question is how wrong do they have to be to not be useful”
Categorizations of Large-Scale Brain Connectivity Analysis

(Bullmore and Sporns, *Nature*, 2009)

**Structural**
- State-invariant, anatomical

**Functional**
- Dynamic, state-dependent, correlative, symmetric

**Effective**
- Dynamic, state-dependent, asymmetric, causal, information flow

**Temporal Scale**
- Hours-Years
- Milliseconds-Seconds
Estimating Functional Connectivity

Popular measures

- Cross-Correlation
- Coherence
- Phase-Locking Value
- Phase-amplitude coupling
- ...

Intro
Theory
SIFT
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Fin
Cross-Correlation and Linear Coherence

**Theory**

Cross-Correlation is a measure of the linear dependence between two variables or signals. It is defined as:

\[ \rho_{AB}(k) = \frac{1}{k} \sum_{j=0}^{k} A(j)B^*(k-j) \]

Linear Coherence (\( \left| C_{AB} \right|^2 \)) can be defined as the magnitude of the cross-correlation function as a function of frequency:

\[ C_{AB}(f) = \sum_{k=0}^{p} \rho_{AB}(k) e^{-i2\pi fk} \]

\[ = \frac{S_{AB}(f)}{\sqrt{S_A(f)S_B(f)}} \]

**Applications**

**Issue:** Linear coherence is biased by auto-power (just as the cross-correlation is biased by strong autocorrelation in individual time series).
Euler’s Formula tells us that any sinusoid can be expressed as the sum of two complex exponentials:

\[ A \cdot \cos(\omega t + \phi) = \frac{A}{2} e^{i(\omega t + \phi)} + \frac{A}{2} e^{-i(\omega t + \phi)} = \text{Re}\{A e^{i(\omega t + \phi)}\} = \text{Re}\{S(\omega, t)\} \]

... or (if real-valued) as the real part of a single complex exponential.

**Phasor**
(Polar Coords)

\[ |S(\omega, t)| = |A| \]

\[ \phi = \angle S(\omega, t) \]

Shorthand notation: \( Ae^{i\phi} \)

Phase shift \( \phi = \pi / 2 \)

Angular frequency \( \omega = 2\pi f = 2\pi \text{ rad/sec} \)

Polar animation courtesy Wikipedia
If we want to examine oscillatory dynamics or relationships between oscillatory signals, analysis in the time domain (i.e. cartesian coordinates) is equivalent to (simpler) operations involving phasors in Fourier space (i.e. polar coordinates).

Polar animation courtesy Wikipedia

(Axis rotated 90° CCW for display)
The Mean Phasor

The average of \( k \) phasors is a new phasor constructed by adding up the original vectors and dividing the length of the resultant vector by \( k \).

If all phasors have similar angles, then vectors will “point” in the same direction and the length of the mean phasor will be comparatively large.

If phasor angles are random, then vectors will point in random directions and the length of the mean phasor will be close to zero.
Phase-Locking Value (PLV)


The main advantage of this approach is that it does not require any a priori hypothesis on the signals. We particular set of data. Our statistical test is based on randomization and is adapted to our known, one must rely on recent techniques of randomization, or bootstrap [Fisher, 1993].

When the sampling distribution of a statistic is unknown, one would detect a phase-locking between the groups. This series are created by shuffling the signals coming from electrode 1. These series are independent of the signals coming from electrode 2, except that we built them to be independent of the signals coming from electrode 2, except that we built which have the same characteristics as the original purpose, we generate 200 new series of variables, values other trials. For each surrogate series $y^{(n)}$, we measure the maximum between $x$ and $y$. The proportion of surrogate values higher than the original PLV (between $x$ and $y$) measures the probability of having false positives for a given level of significance. In this study, we used a criterion of 5% (PLS 5%) to characterize significant synchrony, but this is, of course, a function of the required rigor of significance in the context of the signals being studied. Our method is related to an approach proposed by Friston et al. [1997] to quantify synchrony, but this is, of course, a function of the.

Estimation of phase-locking value. Left: Our synchrony index is designed by shuffling the trials of one of the electrodes (see text for details). By averaging these phase differences across the trials, we directly related to the intertrial variability of the phase differences (PLS $t_0$).

\[ u(t_0) = \frac{1}{N} \sum_{k} e^{i(\phi_2(k,t_0)-\phi_1(k,t_0))} \]

\[ \text{PLV}(t_0) = \text{abs}(u) \]
Phase-Locking Value (PLV)


Computing PLV (“phase coherence”) in EEGLAB:

\texttt{pop_newcrossf(....,'type','phase')}
Phase-Amplitude Coupling

'burst-suppress' oscillators


Local Field Potential (Slow + Fast cells)
PAC may reflect non-stationary or non-linear network dynamics

Graphical Model

Amplitude Modulation

10Hz amplitude coupled to 1 Hz Phase

Time-varying X1→X3 coupling
(1 Hz modulation)
• May present a functional role in execution of cognitive functions (Axmacher et al. 2010; Cohen et al. 2009a,b; Lakatos et al. 2008; Tort et al. 2008, 2009).

• Suggested involvement in **sensory signal detection** (Handel and Haarmeier 2009), **attentional selection** (Schroeder and Lakatos 2009), and **memory processes** (Axmacher et al. 2010; Tort et al. 2009)
Phase-Amplitude Coupling: PLV Method  

- **X_1** original raw signal
- **f_p** filter \(X_1\) at LFO band (e.g. theta)
- **f_A** filter \(X_1\) at HFO band (e.g. gamma)
- \(A_{f_A}\) get amplitude envelope of filtered signal
Phase-Amplitude Coupling: PLV Method


Compute PLV between LFO time-series ($f_p$) and amplitude envelope of HFO time-series ($A_{fA}$).

Significant PLV indicates that the central frequency of $f_p$ modulates the amplitude of the central frequency of $f_A$.

$$\text{PLV}(t_0) = \text{abs}(u(t_0))$$

$$\text{arg}(u(t_0))$$

$$u(t_0) = \frac{1}{N} \sum_{k} e^{i(\phi_{A_{fA}}(k,t_0) - \phi_{f_p}(k,t_0))}$$

PLV across other trials AVERAGE difference phasors across trials
Phase-Amplitude Coupling: PLV Method

**Problem:**

PLV is invariant to differences in amplitude between the two time-series (it only considers phase). Thus PLV-PAC doesn’t take into account the *amplitude* of the co-modulation.

In the example below, $X_1$ and $X_2$ both would produce the same PAC, even though the high-frequency amplitude of $X_2$ clearly is more strongly modulated by the low-frequency rhythm.
Phase-Amplitude Coupling: Modulation Index Method

Original raw signal

Filter X₁ at LFO band (e.g. theta)

Extract the instantaneous phase of fₚ

Filter X₁ at HFO band (e.g. gamma)

Get amplitude envelope of filtered signal
Phase-Amplitude Coupling: Modulation Index Method


$z_i(t_0) = A_{f_A} e^{i\phi_{f_p}}$

$A_{f_A}(t_0) = A_{f_A} \arg(u)$

mean complex phasor $u(t_0)$

Comparison: PLV-PAC

$u(t_0) = \frac{1}{N} \sum_k^N z_k(t_0)$

PAC(t_0) = abs(u)
Phase-Amplitude Coupling: Modulation Index Method


Computing PAC in EEGLAB:

\[
pac(IC1, IC2, ..., 'method', 'mod')
\]

PAC can also be applied *between* sources/channels (e.g. determine whether the phase of oscillation at freq. \(w_p\) in IC1 modulates the amplitude of oscillation at freq. \(w_A\) in IC2. This leads to a measure of cross-frequency (non-linear) functional connectivity.

For Modulation Index method (other modes also available)

Also see PACT plugin for EEGLAB by Miyakoshi et al
(http://sccn.ucsd.edu/wiki/PACT)
(Cross)-Correlation $\neq$ Causation

Coherence/CC/PLV indicate functional, but not effective connectivity
Estimating Effective Connectivity

Non-Invasive

- **Post-hoc** analyses applied to measured neural activity
- **Confirmatory**
  - Dynamic Causal Models
  - Structural Equation Models
- **Exploratory**
  - Granger-Causal methods

- **Data-driven**
- Rooted in *conditional predictability*
- **Scalable** (Valdes-Sosa, 2005)
- Extendable to nonlinear and/or non-stationary systems (Freiwald, 1999; Ding, 2001; Chen, 2004; Ge, 2009)
- Extendable to non-parametric representations (Dhamala, 2009a,b)
- Can be (partially) controlled for (unobserved) exogenous causes (Guo, 2008a,b; Ge, 2009)
- Equivalent to Transfer Entropy for Gaussian Variables (Seth, 2009)
- Flexibly allows us to examine *time-varying* (dynamic) multivariate causal relationships in either the *time* or *frequency* domain
Linear Dynamical Systems

Stochastic Linear Dynamical System

\( X_1(t) = a(t)_{11} X_1(t-1) + a(t)_{12} X_2(t-1) + \epsilon_1(t) \)
\( X_2(t) = a(t)_{22} X_2(t-1) + a(t)_{21} X_1(t-1) + \epsilon_2(t) \)

Order 1 Markov Process (VAR[1])

\( X_1(t) = a(t)_{11} X_1(t-1) + a(t)_{12} X_2(t-1) + \epsilon_1(t) \)
\( X_2(t) = a(t)_{22} X_2(t-1) + a(t)_{21} X_1(t-1) + \epsilon_2(t) \)

\[ \Delta t = 1 \]

Time step
Vector Autoregressive (VAR / MAR / MVAR) Modeling

EEG

\[ X_1(t) \]
\[ X_2(t) \]
\[ \vdots \]
\[ X_M(t) \]

\[ \text{VAR} \]

\[ \text{Granger Causality} \]
\[ \text{Coherence} \]
\[ \text{Spectrum} \]
\[ \ldots \]
VAR Modeling: Assumptions

- “Weak” stationarity of the data
  - mean and variance do not change with time
  - An EEG trace containing prominent evoked potentials is a classic example of a non-stationary time-series

- Stability
  - All eigenvalues of the system matrix are \( \leq 1 \)
  - A stable process will not “blow up” (diverge to infinity)
  - A stable model is always a stationary model (however, the converse is not necessarily true). If a stable model adequately fits the data (white residuals), then the data is likewise stationary
The Linear VAR Model

\[ X(t) = \sum_{k=1}^{p} A^{(k)}(t)X(t-k) + E(t) \]

M-channel data vector at current time \( t \)

M \times M matrix of (possibly time-varying) model coefficients indicating variable dependencies at lag \( k \)

Ordinary Least-Squares
Lattice Filters
Kalman Filtering
Bayesian Methods
Sparse methods
...

\[ A^{(k)}(t) = \begin{pmatrix} a_{11}^{(k)}(t) & \cdots & a_{1M}^{(k)}(t) \\ \vdots & \ddots & \vdots \\ a_{M1}^{(k)}(t) & \cdots & a_{MM}^{(k)}(t) \end{pmatrix} \]

\[ E(t) = N(0, V) \]
Selecting a VAR Model Order

- Model order is typically determined by minimizing information criteria such as Akaike Information Criterion (AIC) for varying model order (p):

\[ \text{AIC}(p) = 2\log(\det(V)) + M^2 p/N \]

Penalizes high model orders (parsimony)

entropy rate (amount of prediction error)

optimal order

![Graph showing AIC values for different model orders]

The graph illustrates the AIC values for varying model orders, with the optimal order marked by a blue dot.
# Model Order Selection Criteria

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<tr>
<th>Estimator</th>
<th>Formula</th>
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<td><strong>Schwarz-Bayes Criterion</strong></td>
<td>$SBC(p) = \ln \left</td>
</tr>
<tr>
<td>(Bayesian Information Criterion)</td>
<td></td>
</tr>
<tr>
<td><strong>Akaike Information Criterion</strong></td>
<td>$AIC(p) = \ln \left</td>
</tr>
<tr>
<td><strong>Akaike’s Final Prediction Error</strong></td>
<td>$FPE(p) = \left</td>
</tr>
<tr>
<td>and its logarithm (used in SIFT)</td>
<td></td>
</tr>
<tr>
<td>$\ln(FPE(p)) = \ln \left</td>
<td>\hat{\Sigma}(p) \right</td>
</tr>
<tr>
<td><strong>Hannan-Quinn Criterion</strong></td>
<td>$HQ(p) = \ln \left</td>
</tr>
</tbody>
</table>

- **More Conservative**
- **Less Conservative**
- **Intermediate Conservative**
Model Order Selection Criteria

Mean Info. Criteria across sampled windows
Optimal order determined by min of mean curve

True Model Order

More Conservative
Intermediate Conservative
Less Conservative
Selecting a VAR Model Order

- Other considerations:
  - A $M$-dimensional VAR model of order $p$ has at most $Mp/2$ spectral peaks distributed amongst the $M$ variables. This means we can observe at most $p/2$ peaks in each variables’ spectrum (or in the cross spectrum between each pair of variables).
  - Optimal model order depends on sampling rate. Higher sampling rate often requires higher model orders.
Model Validation

- If a model is poorly fit to data, then few, if any, inferences can be validly drawn from the model. There a number of criteria which we can use to determine whether we have appropriately fit our VAR model. Here are three commonly used categories of tests:

  - **Whiteness Tests:** checking the residuals of the model for serial and cross-correlation
  - **Consistency Test:** testing whether the model generates data with same correlation structure as the real data
  - **Stability Test:** checking the stability/stationarity of the model.

We’ll discuss these further in Part 2 (Sunday a.m.)
Granger Causality

- First introduced by Wiener (1958). Later reformulated by Granger (1969) in the context of linear stochastic autoregressive models
- Relies on two assumptions:

<table>
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<th>Granger Causality Axioms</th>
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<tbody>
<tr>
<td>1. Causes should precede their effects in time (Temporal Precedence)</td>
</tr>
<tr>
<td>2. Information in a cause’s past should improve the prediction of the effect, above and beyond the information contained in past of the effect (and other measured variables)</td>
</tr>
</tbody>
</table>
Granger Causality

Does $X_4$ granger-cause $X_1$?
(conditioned on $X_2, X_3$)

$X(t) = \sum_{k=1}^{p} A^{(k)} X(t-k) + E(t)$

$X_{-4}(t) = \sum_{k=1}^{p} \tilde{A}^{(k)} X_{-4}(t-k) + \tilde{E}(t)$

prediction error for $X_1$
(variance of residuals $E_1$)
Granger Causality

Granger (1969) quantified this definition for **bivariate** processes in the form of an F-ratio:

\[
F_{X_1 \leftarrow X_2} = \ln \left( \frac{\text{var}(\tilde{E}_1)}{\text{var}(E_1)} \right) = \ln \left( \frac{\text{var}(X_1(t) \mid X_1(\cdot))}{\text{var}(X_1(t) \mid X_1(\cdot), X_2(\cdot))} \right)
\]

Alternately, for a **multivariate interpretation** we can fit a single VAR model to all channels and apply the following definition:

**Definition 1**

\[X_j \text{ granger-causes } X_i \text{ conditioned on all other variables in } X\]

**if and only if** \[A_{ij}(k) \gg 0\] **for some lag** \(k \in \{1, \ldots, p\}\)
Granger Causality Quiz

Example: 2-channel VAR process of order 1

\[
\begin{pmatrix}
X_1(t) \\
X_2(t)
\end{pmatrix} =
\begin{pmatrix}
-0.5 & 0 \\
0.7 & 0.2
\end{pmatrix}
\begin{pmatrix}
X_1(t-1) \\
X_2(t-1)
\end{pmatrix} +
\begin{pmatrix}
E_1(t) \\
E_2(t)
\end{pmatrix}
\]

Which causal structure does this model correspond to?

a) 1 → 2  
   b) 1 ← 2  
   c) 1 ←→ 2
Granger Causality – Frequency Domain

\[ X(t) = \sum_{k=1}^{P} A^{(k)} X(t - k) + E(t) \]

Fourier-transforming \( A^{(k)} \) we obtain

\[ A(f) = -\sum_{k=0}^{P} A^{(k)} e^{-i2\pi fk}; A^{(0)} = I \]

Likewise, \( X(f) \) and \( E(f) \) correspond to the fourier transforms of the data and residuals, respectively.

We can then define the spectral matrix \( X(f) \) as follows:

\[ X(f) = A(f)^{-1} E(f) = H(f)E(f) \]

Where \( H(f) \) is the transfer matrix of the system.

**Definition 2**

\[ X_i \text{ granger-causes } X_i \text{ conditioned on all other variables in } X \]

if and only if \( |A_{ij}(f)| \gg 0 \) for some frequency \( f \)
\[ X(t) = \sum_{k=1}^{p} A^{(k)}(t) X(t-k) + E(t) \]
\[ A(f,t) = -\sum_{k=0}^{p} A^{(k)}(t)e^{-\frac{i\pi}{2} f k}; \quad A^{(0)} = I \]
\[ X(f,t) = A(f,t)^{-1} E(f,t) = H(f,t) E(f,t) \]

\[ S(f) = X(f)X(f)^* \]
\[ = H(f)\Sigma H(f)^* \]

\[ C_{ij}(f) = \frac{S_{ij}(f)}{\sqrt{S_{ii}(f)S_{jj}(f)}} \]
\[ (Bendat and Piersol, 1986) \]

\[ F_{ij}(f) = \frac{\Sigma_{ij} - (\Sigma_{ij}^2 / \Sigma_{ii}) |H_{ij}(f)|^2}{S_{ii}(f)} \]
\[ (Geweke, 1982; Bressler et al., 2007) \]

\[ P_{ij}(f) = \frac{S_{ij}^{-1}(f)}{\sqrt{S_{ii}^{-1}(f)S_{jj}^{-1}(f)}} \]
\[ (Bendat and Piersol, 1986; Dalhaus, 2000) \]

\[ \pi_{ij}^2(f) = \frac{|A_{ij}(f)|^2}{\sum_{k=1}^{M} |A_{kj}(f)|^2} \]
\[ (Baccalà and Sameshima, 2001) \]
<table>
<thead>
<tr>
<th>Estimator</th>
<th>Formula</th>
</tr>
</thead>
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<tr>
<td>Spectral Density Matrix</td>
<td>$S(f) = X(f)X(f)^*$</td>
</tr>
<tr>
<td></td>
<td>$= H(f)\Sigma H(f)^*$</td>
</tr>
<tr>
<td>Coherency</td>
<td>$C_\gamma(f) = \frac{S_\gamma(f)}{\sqrt{S_\alpha(f)S_\beta(f)}}$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
</tr>
<tr>
<td>Imaginary Coherence (iCoh)</td>
<td>$iCoh_\gamma(f) = \text{Im}(C_\gamma(f))$</td>
</tr>
<tr>
<td>Partial Coherence (pCoh)</td>
<td>$P_\gamma(f) = \frac{\hat{S}<em>\gamma(f)}{\sqrt{\hat{S}</em>\alpha(f)\hat{S}_\beta(f)}}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{S}(f) = S(f)^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
</tr>
<tr>
<td>Multiple Coherence (mCoh)</td>
<td>$G_\gamma(f) = \sqrt{1 - \text{det}(S(f))}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{S_\gamma(f)\mathbf{M}_\gamma(f)}$</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{M}_\gamma(f)$ is the minor of $S(f)$ obtained by removing the $i^{th}$ row and column of $S(f)$ and returning the determinant.</td>
</tr>
<tr>
<td>Normalized Partial Directed Coherence (PDC)</td>
<td>$\pi_\gamma(f) = \frac{A_\gamma(f)}{\sqrt{\sum_{i=1}^M</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
</tr>
<tr>
<td></td>
<td>$\sum_{i=1}^M</td>
</tr>
<tr>
<td>Generalized Partial Directed Coherence (GPDC)</td>
<td>$\mathbf{\pi}<em>\gamma(f) = \frac{1}{\sum</em>{i=1}^M</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
</tr>
<tr>
<td></td>
<td>$\sum_{i=1}^M</td>
</tr>
<tr>
<td>Renormalized Partial Directed Coherence (rPDC)</td>
<td>$\lambda_\gamma(f) = Q_\gamma(f)\mathbf{V}<em>\gamma(f)^{-1}\mathbf{Q}</em>\gamma(f)$</td>
</tr>
<tr>
<td></td>
<td>where $Q_\gamma(f) = \begin{pmatrix} \text{Re}[A_\gamma(f)] \ \text{Im}[A_\gamma(f)] \end{pmatrix}$ and $\mathbf{V}<em>\gamma(f) = \sum</em>{k,l} R_{\gamma}^{-1}(k,l) \Sigma_{\gamma} Z(2\pi f, k,l) Z(\alpha, k,l)$</td>
</tr>
<tr>
<td></td>
<td>$Z(\alpha, k,l) = \begin{pmatrix} \cos(\alpha k) \cos(\alpha l) \ \cos(\alpha k) \sin(\alpha l) \ \sin(\alpha k) \cos(\alpha l) \ \sin(\alpha k) \sin(\alpha l) \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$\mathbf{R}$ is the $[(M_p)^2 \times (M_p)^2]$ covariance matrix of the VAR$p$ process (Lütkepohl, 2006)</td>
</tr>
<tr>
<td>Granger-Geweke Causality (GGC)</td>
<td>$F_\gamma(f) = \frac{(\Sigma_\gamma - (\Sigma_\gamma/\Sigma_\gamma))</td>
</tr>
</tbody>
</table>

**Transfer Function**

$H(f)$

**System Matrix**

$A(f)$

**Noise Covariance Matrix**

$\Sigma$

**Variance Stabilization**

$\eta_\gamma(f) = \frac{|H_\gamma(f)|^2}{\sum_{i=1}^M |H_i(f)|^2}$

$0 \leq |\eta_\gamma(f)|^2 \leq 1$

$\sum_{i=1}^M |\eta_i(f)|^2 = 1$

$\delta_\gamma(f) = \eta_\gamma(f)P_\gamma(f)$

$\mathbf{X}(t) = \sum_{k=1}^p \mathbf{A}^{(k)}(t)\mathbf{X}(t-k) + \mathbf{E}(t)$

$\mathbf{A}(f, t) = -\sum_{k=0}^p \mathbf{A}^{(k)}(t)e^{-\mathbf{i}2\pi ft}$; $\mathbf{A}^{(0)} = I$

$\mathbf{X}(f, t) = \mathbf{A}(f, t)^{-1}\mathbf{E}(f, t) = \mathbf{H}(f, t)\mathbf{E}(f, t)$

For additional details, see SIFT Handbook (sccn.ucsd.edu/wiki/SIFT)
Scalp or Source?

Fig. 15.13: Direction of flows for 21-channel EEG (awake state eyes closed) obtained by means of different methods. The shade of gray of the arrow represents the strength of the connection (black = the strongest), for each method 40 strongest flows are shown. Reprinted from with permission [49] (© IEEE 2005).

a lot of activity flowing to the destination channels from the posterior electrodes, so the denominator in Eq. (15.6) is quite large, which diminishes the values of DTFs showing outflows from Fz. For Granger causality and DTF there is no propagation from the temporal electrodes. This is practically also the case for dDTF. The dDTF shows only direct flows, we can see that in this case the pattern of flows is consistent with anatomy, e.g., a lack of direct connection between Oz and Pz, Fz, and Fpz—locations where hemispheres are partitioned. The main sources of the activity—namely, electrodes P3, P4, O2, Oz, O1—are the same as for the other multivariate estimates.

Inspecting the results of application of the PDC function to the same data epoch we observe a different picture. One can notice that, unlike the results of dDTF, some channels became sinks. This is due to the normalization of PDC. In fact, we do not see the transmission, as is the case for dDTF, but the ratio between the flow to a given channel with respect to all the outflows from the considered channel. In this way, a channel propagating activity in all directions will show weaker flows than a channel propagating only in one direction. Therefore, the method is not suitable for identification of sources of EEG activity, but it may be useful when the destination channel is of primary interest.
Scalp or Source?

\[ X(t) = HS(t) = \sum_{k=1}^{p} HA^{(k)}(t) H^{-1} X(t - k) + HE(t) \]

\[ X(t) = HS(t) \]

Solution? Source Separation

\[ S(t) = \sum_{k=1}^{p} A^{(k)}(t) S(t - k) + E(t) \]
Forward/Inverse Modeling

A Recipe for Reducing Errors:
- Realistic Forward Model
- Appropriately Constrained Inverse Model

NFT

ill-posed!
solutions?
impose constraints!
sparse/smooth independence anatomy ...

Forward Modeling

$S(t) = MS(t)$

Inverse Modeling

$	ilde{S}(t) = M^{-1} X(t)$

Akalin Acar, et al, 2010

Theory
### Forward/Inverse Modeling

<table>
<thead>
<tr>
<th>Method</th>
<th>Smoothness</th>
<th>Sparsity</th>
<th>Independence/Orthogonality</th>
</tr>
</thead>
<tbody>
<tr>
<td>MNE</td>
<td>X</td>
<td></td>
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</tr>
<tr>
<td>LORETA</td>
<td>X</td>
<td></td>
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<tr>
<td>dSPM</td>
<td>X</td>
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<tr>
<td>Beamforming</td>
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<td>X</td>
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<tr>
<td>Sparse Bayesian Learning</td>
<td>X</td>
<td>X</td>
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<tr>
<td>S-FLEX</td>
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<td>X</td>
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<tr>
<td>FOCUSS</td>
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<td>X</td>
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<tr>
<td>ICA/PCA/SOBI</td>
<td></td>
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<td>X</td>
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</tbody>
</table>

Source reconstruction with ICA+SBL

[Simulated](#) [Reconstructed](#) [Error](#)

![Brain Images](image)

Isn’t it a contradiction to examine dependence between Independent/Uncorrelated Components?

Instantaneous (e.g., Infomax) ICA only explicitly seeks to maximize instantaneous independence. Time-delayed dependencies may be preserved.

Infomax ICA seeks to maximize global independence (over entire recording session), transient dependencies may be preserved.

Independence is a very strict criterion that cannot be achieved in general by a linear transformation (such as ICA). Instead, dependent variables will form a dependent subspace.

However, the best approach is to use an inverse model that explicitly preserves time-delayed dependencies or jointly estimates sources (de-mixing matrix) and connectivity (VAR parameters). See Haufe, 2008 IEEE TBME for a good treatment (coming soon to SIFT).
Estimating Dependency of Independent Components

Let us recall the assumptions we make to identify individual sources. However, for SCSA there is still room for improvement. The goodness of fit is in the medium range, and CICAAR requires the longest time. While the EM implementation of SCSA is relatively short, the other approaches are significantly slower. To-D0 Theory

<table>
<thead>
<tr>
<th>Connectivity Error</th>
<th>Connectivity Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>N0</td>
<td>N1</td>
</tr>
<tr>
<td>N2</td>
<td>N3</td>
</tr>
<tr>
<td>N4</td>
<td>N5</td>
</tr>
<tr>
<td>N6</td>
<td></td>
</tr>
</tbody>
</table>

Mixing Matrix Approximation Error:

N1 = 1.5
N2 = 2.0
N3 = 2.5
N4 = 3.0
N5 = 3.5
N6 = 4.0

Dependence on Connectivity and SNR:

SNR = 0, 0.2, 0.4, 0.6

Results are shown for the proposed Sparsely Connected Sources Analysis variants and three alternative approaches: SCSA, CSA, and three alternative approaches: MVARICA, CICAAR, and ICA.

Haufe et al, IEEE TBME 2008
Adapting to Non-Stationarity

- The brain is a **dynamic system** and measured brain activity and coupling can change rapidly with time (non-stationarity)
  - event-related perturbations (ERSP, ERP, etc)
  - structural changes due to learning/feedback

- **How can we adapt to non-stationarity?**

![Graph showing a line with peaks and troughs representing mV over time.](image)
Adapting to Non-Stationarity

- Many ways to do adaptive VAR estimation
- Two popular approaches (adopted in SIFT):
  - Segmentation-based adaptive VAR estimation (assumes local stationarity)
  - State-Space Modeling
Adapting to Non-Stationarity

- Many ways to do adaptive VAR estimation
- Two popular approaches (adopted in SIFT):
  - Segmentation-based adaptive VAR estimation (assumes local stationarity)
  - State-Space Modeling
Segmentation-based VAR

(Jansen et al., 1981; Florian and Pfurtscheller, 1995; Ding et al., 2000)

\[ X(t) = \sum_{k=1}^{p} A^{(k)}(t)X(t-k) + E(t) \]

\[ A(f,t) = -\sum_{k=0}^{p} A^{(k)}(t)e^{-i2\pi fk}; A^{(0)} = I \]

Analogous to short-time Fourier transform
INTERMISSION