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# SOLUTION OF DIFFERENTIAL EQUATIONS OF HYPERGEOMETRIC TYPE

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ABSTRACT. We present a method for solving the classical linear ordinary differential equations of hypergeometric type [8], including Bessel's equation, Legendre's equation, and others with polynomial coefficients of a certain type. The method is characterized by using the Mellin transform to convert the original differential equation into a complex difference equation solving the differential equations in the Mellin transform, or log-spectral, domain.

## 1. INTRODUCTION

The standard method of solving differential equations with variable coefficients is the series method of Frobenius. If certain order criteria apply to the singularities, then a series solution can be generated. Two generate an independent

This paper considers the solution of differential equations of hypergeometric type. The proposed method is based on using the Mellin transform to convert the differential equation into a first order difference equation in the complex domain, for which solutions can be constructed using Γ-functions. We propose that the common feature of functions of hypergeometric type[1] is this property of yielding a first order complex difference equation. The Mathieu equation, for example, yields a second order difference equation, which is not solvable by the proposed method. This corresponds to its being of "higher type" than the functions of hypergeometric type [8, §19.1].

An advantage of the proposed method over series methods like that of Frobenius, is that multiple linearly independent solutions are found simultaneously, and may be represented in the original domain either in series form, or as the scale convolution of two or more constituent functions. A notable case in which a second solution cannot be found using the Frobenius method is Bessel's equation of integer order. Bessel's function of the second kind is commonly *defined* and then *verified* to be an independent solution and satisfy Bessel's equation. Similarly for Legendre's function of the second kind. Our method yields the second solutions, plus others, simultaneously with the first. Furthermore, each solution is represented in a schematic form that can be seen as being central in the sense that and

Our method differs from that used by Mellin [**?**, 3, §7.6] in our emphasis on the difference equation and the simultaneous generation of multiple independent solutions for various boundary conditions. The differen [2, §7.6] by Hille . Hille, however, seems to treat the method more as a peculiarity [2, p. 279] than a general method, using the Frobenius method in the main treatment of differential equations.

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[1] There is evidently no standard definition [4, p. 56] of hypergeometric type.

Again, Bessel's equation of the second kind is *defined* and *verified* to be a solution of Bessel's equation, rather than derived on an equal footing with other solutions.

In § 2, we review the basic properties and inverse theorems for the Mellin transform. In § 3 we demonstrate the method for solving some classical equations.

## 2. The Mellin transform

The Mellin transform [7, 5, 1], is defined by,

$$\mathcal{M}\left[f(x)\,;s\right] \;\equiv\; \tilde{f}(s) \;\equiv\; \int_0^\infty x^{s-1} f(x)\,dx \tag{1}$$

for $s \in \mathbb{C}$ such that the integral is convergent.

2.1. **Basic properties.** If $x^k g(x)$ and $x^k h(x) \in L(0,\infty)$ for some $k \in \mathbb{R}$, then [7, Thm. 44],

$$f(x) = \int_0^\infty \frac{1}{\xi}\, g\!\left(\frac{x}{\xi}\right) h(\xi)\,d\xi \quad\Rightarrow\quad \tilde{f}(s) = \tilde{g}(s)\tilde{h}(s) \tag{2}$$

and $x^k f(x) \in L(0,\infty)$. Thus under appropriate conditions, we can solve for $h(x)$ by inverting the transform. If $x^k f(x) \in L(0,\infty)$, then (1) can be inverted almost everywhere using the formula [7, Thm. 28],

$$\mathcal{M}^{-1}[\tilde{f}(s)\,;x] \;\equiv\; \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} x^{-s}\tilde{f}(s)\,ds \;=\; \frac{f(x+)+f(x-)}{2} \tag{3}$$

where $f(x+)$ and $f(x-)$ denote the right and left hand limits of $f$ at $x$. Thus, for example, when $h$ in (2) is continuous on $(0,\infty)$, we have,

$$h(x) = \mathcal{M}^{-1}\Big[\frac{\tilde{f}(s)}{\tilde{g}(s)}\,;x\Big], \quad x \in (0,\infty)$$

We can similarly solve integral equations of the form $f(x) = \int_0^\infty g(\xi x)h(\xi)\,d\xi$. Using the following two properties, which follow readily from the definition (1),

$$\mathcal{M}\left[x^a f(x)\,;s\right] = \frac{1}{a}\tilde{f}\!\left(\frac{s}{a}\right), \quad \mathcal{M}\left[f(x^a)\,;s\right] = \tilde{f}(s+a) \tag{4}$$

we have $\mathcal{M}\left[t^{-1}h(t^{-1})\,;s\right] = \tilde{h}(1-s)$, and it follows that,

$$f(x) = \int_0^\infty g(\xi x)h(\xi)\,d\xi \quad\Rightarrow\quad \tilde{f}(s) = \tilde{g}(s)\tilde{h}(1-s) \tag{5}$$

Like the Laplace transform, the Mellin transform can be used to convert integro-differential equations into algebraic equations. We use in particular the following relation. Let $D$ denote the differential operator. The transform of the operator $(-x)^n D^n$ is given by,

$$\mathcal{M}\left[(-x)^n D^n f(x)\,;s\right] = \frac{\Gamma(s+n)}{\Gamma(s)}\tilde{f}(s) \tag{6}$$

There are two basic integrals that are used, $\int_0^x f(x)dx$ and $\int_x^\infty$, and there are two corresponding definitions of fractional integrals. The *Riemann-Liouville fractional integral* [1, p. 113], for $\alpha > 0$ and non-integral, is defined by,

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (t-x)^{\alpha-1} f(t)\,dt$$

The *Riemann-Liouville fractional derivative*, $D^{\beta}$, $\beta > 0$, is defined by the same formula, with $\alpha$ replaced by $-\beta$.

The *Weyl fractional integral*, for $\alpha > 0$ and non-integral, is defined by,

$$W^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\, dt$$

For the *Weyl fractional derivative*, $W^{\beta}$, $\beta > 0$, let $n$ be the smallest integer greater than $\beta$. Then the Weyl fractional derivative is defined by,

$$W^{\beta} f(x) = (-D)^n\, W^{n-\beta} f(x)$$

The Mellin transform of the Weyl fractional integral or derivative ($\alpha$ positive or negative), is given by,

$$\mathcal{M}\left[W^{-\alpha} f(x)\,;s\right] = \frac{\Gamma(s)}{\Gamma(s+\alpha)}\, \tilde{f}(s+\alpha) \tag{7}$$

We also use the following result on differentiation in the transform domain.

$$\mathcal{M}\left[(\log x)^n f(x)\,;s\right] \;=\; \frac{d^n}{ds^n}\, \tilde{f}(s) \tag{8}$$

2.2. **Inversion theorems.** When $\tilde{f}(\sigma + it)$ decays exponentially as $|t| \to \infty$, the Mellin transform can be inverted using the following theorem [7, Thm. 31],

**Theorem 1.** *Let $s = \sigma + it$ and $z = re^{i\theta}$. The function $\tilde{f}(s)$ is given by,*

$$\tilde{f}(s) = \int_0^{\infty} z^{s-1} f(z)\, dz$$

*and has the following two properties,*

- (S1) *$\tilde{f}(s) = \tilde{f}(\sigma + it)$ is analytic in the strip, $a < \sigma < b$,*
- (S2) *$|\tilde{f}(\sigma + it)| = O(e^{(\alpha+\epsilon)t})$ as $t \to -\infty$, and $O(e^{-(\beta-\epsilon)t})$ as $t \to \infty$, (with $0 < \alpha, \beta \leq \pi$) uniformly in any closed strip interior to $a < \sigma < b$,*

*if and only if, the function $f(z)$ is given by,*

$$f(z) = \int_{k-i\infty}^{k+i\infty} z^{-s} \tilde{f}(s)\, ds$$

*for any $a < k < b$, and has the two properties,*

- (Z1) *$f(z) = f(re^{i\theta})$ is analytic in the wedge $-\alpha < \theta < \beta$,*
- (Z2) *$|f(re^{i\theta})| = O(r^{-a-\epsilon})$ as $r \to 0$, and $O(r^{-b+\epsilon})$ as $r \to \infty$, uniformly in any closed wedge interior to $-\alpha < \theta < \beta$ .*

If $\tilde{f}(s)$ does not decay exponentially as $|t| \to \infty$, then we require another theorem for the inverse.

**Theorem 2.** *Let the integral $\int_0^{\infty} x^{s-1} f(x)\, dx \equiv \tilde{f}(s)$ be uniformly convergent (as the limits are approached independently) for $s = k + it$, $t$ in any finite interval. Then,*

$$\frac{1}{2\pi i} \lim_{\lambda\to\infty} \int_{k-i\lambda}^{k+i\lambda} \left(1 - \frac{|t|}{\lambda}\right) x^{-s} \tilde{f}(s)\, ds \;=\; \frac{f(x+) + f(x-)}{2} \tag{9}$$

*for all $x > 0$ such that $f(x+)$ and $f(x-)$ exist.*

The asymptotic order of the Gamma function can be determined from Stirling's formula. Specifically [6, §4.42], for fixed $\theta$, we have,

$$\begin{aligned}\left|\Gamma(re^{i\theta})\right| &= O\big(\left|z^{z-\frac{1}{2}}e^{-z}\right|\big)\\ &= O\big(r^{r\cos\theta-\frac{1}{2}}e^{-r\cos\theta-r\,\theta\sin\theta}\big)\end{aligned}$$

as $r \to \infty$. And for fixed $\sigma$,

$$\begin{aligned}\left|\Gamma(\sigma+it)\right| &= O\big(\left|s^{s-\frac{1}{2}}e^{-s}\right|\big)\\ &= O\big(|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\big)\end{aligned}$$

as $|t| \to \infty$, and,

## 3. Differential equations of hypergeometric type

**3.1. Classical equations.** Bessel's differential equation is defined by,

$$x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + (x^2-\nu^2)f = 0 \tag{10}$$

Taking Mellin transforms, we have,

$$s(s+1)\tilde{f}(s) - s\tilde{f}(s) + \tilde{f}(s+2) - \nu^2\tilde{f}(s) = 0$$

or, rearranging,

$$\tilde{f}(s+2) = (\nu^2 - s^2)\tilde{f}(s) \tag{11}$$

That is, when $\tilde{f}(s)$ is shifted to the left by 2, two simple zeros are introduced, one at $\nu$ and one at $-\nu$.

Now consider the hypergeometric differential equation, which is defined by,

$$x(1-x)\frac{d^2f}{dx^2} + \big(c - (a+b+1)\,x\big)\frac{df}{dx} + abf = 0$$

Taking Mellin transforms and rearranging, we get,

$$\tilde{f}(s-1) = \frac{(s-a)(s-b)}{(s-c)(s-1)}\,\tilde{f}(s) \tag{12}$$

Here, shifting $\tilde{f}(s)$ to the right by 1 introduces zeros at $a$ and $b$, and poles at $c$ and 1. The following table lists the classical second order differential equations that give rise to first order complex difference equations,

| **Name** | **Differential Equation** | **Interval** |
|---|---|---|
| Bessel | $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ | $(0, \infty)$ |
| Modified Bessel | $x^2y'' + xy' - (x^2 + \nu^2)y = 0$ | $(0, \infty)$ |
| Spherical Bessel | $x^2y'' + 2xy' + (x^2 - \nu(\nu+1))y = 0$ | $(0, \infty)$ |
| Struve | $x^2y'' + xy' + (x^2 - \nu^2)y = \frac{2^{1-\nu}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\, x^{\nu+1}$ | $(0, \infty)$ |
| Legendre | $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ | $(-1, 1)$ |
| Chebyshev I | $(1-x^2)y'' - xy' + n^2y = 0$ | $(-1, 1)$ |
| Chebyshev II | $(1-x^2)y'' - 3xy' + n(n+2)y = 0$ | $(-1, 1)$ |
| Laguerre | $xy'' - (1-x)y' + ny = 0$ | $(0, \infty)$ |
| Confluent hypergeometric | $xy'' + (c-x)y' - ay = 0$ | $-$ |
| Hermite | $y'' - 2xy' + 2ny = 0$ | $(-\infty, \infty)$ |
| Hypergeometric | $x(1-x)y'' + (c-(a+b+1)x)y' + ab\,y = 0$ | $-$ |

The corresponding Mellin difference equations are given in the following table.

| **Type** | **Complex difference equation** |
|---|---|
| Bessel | $\tilde{y}(s+2) = \pm(s-\nu)(s+\nu)\tilde{y}(s)$ |
| Legendre | $\tilde{y}(s-2) = \frac{(s-n)(s-(n+1))}{(s-1)(s-2)}\,\tilde{y}(s)$ |
| Confluent hypergeometric | $\tilde{y}(s-1) = \frac{(s-a)}{(s-c)(s-1)}\,\tilde{y}(s)$ |
| Hermite | $\tilde{y}(s-2) = -2\frac{(s+n)}{(s-1)(s-2)}\,\tilde{y}(s)$ |
| Hypergeometric | $\tilde{y}(s-1) = \frac{(s-a)(s-b)}{(s-c)(s-1)}\,\tilde{y}(s)$ |

In each case, we have reduced the problem to the determination of a complex function, integrable in some strip, satisfying,

$$F(s+\delta) = \frac{p(s)}{q(s)}\,F(s) \tag{13}$$

where $p(s)$ and $q(s)$ are polynomials with real zeros. The basic idea of the proposed method is to *construct* meromorphic solutions to (13) by appropriately combining $\Gamma$-functions.

For our purposes, a Γ-function, say $\Gamma((s-a)/b),\ a, b \in \mathbb{R}$, is regarded as an infinite sequence of poles beginning at $a$, extending in the direction $-\mathrm{sgn}(b)$ and evenly spaced at a distance of $|b|$. Similarly, the function $\cos(\pi(s-a)/b)$ is viewed as an infinite sequence of zeros extending in the positive and negative directions. The classical formulae relating the trigonometric and Γ functions are easily deduced by viewing the functions this way and matching poles and zeros.

$$\frac{\pi}{\sin(\pi s)} = \Gamma(s)\Gamma(1-s) \qquad \frac{\pi}{\cos(\pi s)} = \Gamma(\tfrac{1}{2}+s)\Gamma(\tfrac{1}{2}-s)$$

3.2. **Difference equations.** Suppose we wish to construct a function that satisfies,

$$F(s+\delta) = (s-a)F(s) \tag{14}$$

where $\delta \in \mathbb{R}$. That is, shifting $F(s)$ to the left by $\delta$ introduces a simple zero at $a$. The Γ-function is the prototype of functions of this kind, satisfying, $\Gamma(s+1) = s\Gamma(s)$. Let us first note that if $F(s)$ satisfies (14), then

$$F(s)\cos(2\pi(s-r)/\delta) \quad \text{and} \quad \frac{F(s)}{\cos(2\pi(s-r)/\delta)}$$

also satisfy (14) for any $r \in \mathbb{R}$. We can use this type of cosine term to change the asymptotic order of $F(s)$ to make it integrable. Also note that if $F$ satisfies,

$$F(s+\delta) = G(s)F(s) \tag{15}$$

and we define $\bar{F}$ by,

$$\bar{F}(s) \equiv F(s)\cos(\pi(s-r)/\delta)$$

then $\bar{F}$ satisfies,

$$\bar{F}(s+\delta) = -G(s)\bar{F}(s) \tag{16}$$

for any $r \in \mathbb{R}$. This type of cosine term can be used when it is desired to change the sign in the functional equation satisfied by $F(s)$. We shall denote these cosine terms by,

$$\mathsf{C}_1(r) \equiv \cos\big(\pi(s-r)/\delta\big) \qquad \mathsf{C}_2(r) \equiv \cos\big(2\pi(s-r)/\delta\big) \tag{17}$$

The $\delta$ will generally be taken to be understood from the equation being considered, and we assume that the functional equation is shifted to make $\delta$ positive.

As a final preliminary observation, note that if $F(s)$ satisfies (15), and we define,

$$\bar{F}(s) \equiv \alpha^{s/\delta} F(s)$$

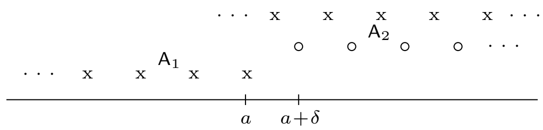
with $\alpha > 0$, then $\bar{F}$ satisfies,

$$\bar{F}(s+\delta) = \alpha\, G(s)\bar{F}(s)$$

Now, in order to construct a function such that a shift introduces a zero at $a$, we have two basic options: either shift the end of a one-sided sequence of zeros *on*to $a$, or shift the end of a one-sided sequence of poles *off* of $a$. Likewise, to introduce a pole at $a$, we can either shift a sequence of poles onto $a$, or shift a sequence of zeros off of $a$. Specifically, to construct a function satisfying (14), we can use either of,

$$\mathsf{A}_1(a) \equiv \delta^{s/\delta}\,\Gamma\big((s-a)/\delta\big) \qquad \mathsf{A}_2(a) \equiv \frac{\delta^{s/\delta}}{\Gamma\big((a+\delta-s)/\delta\big)\cos(\pi(s-r)/\delta)} \tag{18}$$

where the cosine in $\mathsf{A}_2$ is used to turn $a-s$ into $s-a$, as in (16). The following figure illustrates the idea. In the cosine term in $\mathsf{A}_2$, $r$ is taken to be $a$.




When $\mathsf{A}_1$ *or* $\mathsf{A}_2$ is shifted to the left by $\delta$, the effect is to introduce a simple zero at $a$. Similarly, to introduce a simple pole at $a$, so that $F$ satisfies $F(s+\delta) = F(s)/(s-a)$, we can use either of,

$$\mathsf{B}_1(a) \equiv \frac{\delta^{-s/\delta}}{\Gamma\big((s-a)/\delta\big)} \qquad \mathsf{B}_2(a) \equiv \delta^{-s/\delta}\,\frac{\Gamma\big((a+\delta-s)/\delta\big)}{\cos(\pi(s-r)/\delta)} \tag{19}$$

Using these elementary terms, we can easily write down solutions to the Bessel and hypergeometric differential equations in the Mellin domain. Since there are two possible elementary terms corresponding to each zero and pole introduced, if the total number of zeros and poles introduced is $n$, then we can formally write down $2^n$ solutions.

3.3. **Bessel's equation.** The Mellin transform of Bessel's differential equation yields the difference equation,

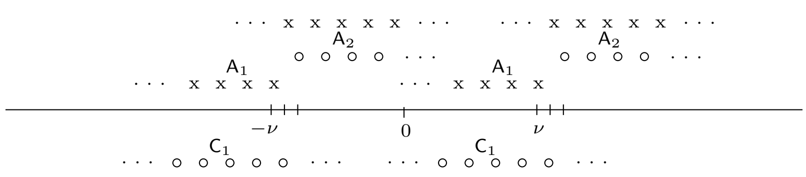
$$\tilde{f}(s+2) = -(s+\nu)(s-\nu)\tilde{f}(s) \tag{20}$$

The elementary terms corresponding to the creation of a zero at $-\nu$ and at $\nu$ are,

$$\mathsf{A}_1(-\nu) = 2^{s/2}\,\Gamma\left(\frac{s+\nu}{2}\right) \quad \mathsf{A}_1(\nu) = 2^{s/2}\,\Gamma\left(\frac{s-\nu}{2}\right) \quad \mathsf{C}_1(\pm\nu) = \cos\left(\frac{\pi(s\mp\nu)}{2}\right)$$

$$\mathsf{A}_2(-\nu) = \frac{2^{s/2}}{\Gamma\left(\frac{-\nu+2-s}{2}\right)\mathsf{C}_1(-\nu)} \quad \mathsf{A}_2(\nu) = \frac{2^{s/2}}{\Gamma\left(\frac{\nu+2-s}{2}\right)\mathsf{C}_1(\nu)}$$

These elements are depicted graphically in the following figure,




where the x's represent poles and the circles represent zeros, and each sequence is on the line $\mathrm{Im}(s) = 0$, though separated vertically in the figure for clarity. Real parts are accurate. We take one of the $-\nu$ terms, one of the $\nu$ terms, and a cosine term, so that when everything is shifted to the left by two, the effect is to introduce zeros at $-\nu$ and $\nu$, and multiply by $-1$. The cosine terms are also chosen to ensure that the function is integrable on some vertical line $\mathrm{Re}(s) = k$, so that the Mellin transform can be inverted. Using (17) to write the cosine expressions in terms of $\Gamma$-functions, we have the following four basic solutions,

$$\mathsf{A}_1\mathsf{A}_1\mathsf{C}_1 = \pi\, 2^s \frac{\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)}{\Gamma\left(\frac{1+s-r}{2}\right)\Gamma\left(\frac{1-s-r}{2}\right)} \qquad \mathsf{A}_2\mathsf{A}_1\mathsf{C}_1 = 2^s \frac{\Gamma\left(\frac{s-\nu}{2}\right)}{\Gamma\left(\frac{-\nu+2-s}{2}\right)}$$

$$\mathsf{A}_1\mathsf{A}_2\mathsf{C}_1 = 2^s \frac{\Gamma\left(\frac{s+\nu}{2}\right)}{\Gamma\left(\frac{\nu+2-s}{2}\right)} \qquad \mathsf{A}_2\mathsf{A}_2\mathsf{C}_1 = \frac{2^s}{\pi} \frac{\Gamma\left(\frac{1+s-r}{2}\right)\Gamma\left(\frac{1-s-r}{2}\right)}{\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)}$$

where $r$ is arbitrary. Each of these expressions is a solution to (20). They will be shown to correspond respectively to $Y_\nu$ (Bessel function of the second kind), $J_\nu$, $J_{-\nu}$ (Bessel functions of the first kind), and $\mathbf{H}_{-\nu}$ (Struve's function of order $-\nu$).

3.3.1. *Series solutions.* Using the theory of residues and contour integration, together with the asymptotic properties of the $\Gamma$-function, we can easily derive series solutions corresponding to each solution to the complex difference equation.

For $\mathsf{A}_1\mathsf{A}_1\mathsf{C}_1$, we have $\tilde{f}(\sigma + it) = O(||)$

According to Theorem 1

3.3.2. *Integral representations.*

3.4. **Hypergeometric equation.** We have seen that the Hypergeometric differential equation gives rise, after rearranging (12), to the following quasi-periodic functional equation,

$$\tilde{f}(s+1) = \frac{s\big(s-(c-1)\big)}{\big(s-(a-1)\big)\big(s-(b-1)\big)}\,\tilde{f}(s) \tag{21}$$

Here $\delta$ is 1, and the elementary terms are,

$$\begin{aligned} \mathsf{A}_1(0) &= \Gamma(s) & \mathsf{A}_2(0) &= \frac{1}{\Gamma(1-s)\mathsf{C}_1} \\ \mathsf{A}_1(c-1) &= \Gamma(s+1-c) & \mathsf{A}_2(c-1) &= \frac{1}{\Gamma(c-s)\mathsf{C}_1} \\ \mathsf{B}_1(a-1) &= \frac{1}{\Gamma(s+1-a)} & \mathsf{B}_2(a-1) &= \Gamma(a-s)\mathsf{C}_1 \\ \mathsf{B}_1(b-1) &= \frac{1}{\Gamma(s+1-b)} & \mathsf{B}_2(b-1) &= \Gamma(b-s)\mathsf{C}_1 \end{aligned}$$

where $\mathsf{C}_1$ indicates some kind of cosine term of the form $\cos(\pi(s-r))$ or $1/\cos(\pi(s-r))$. The purpose of these terms with respect to the functional is to multiply by $-1$, so any two of the $\mathsf{C}_1$ terms multiplied or divided can be chosen to cancel each other out, so long as the resulting function is invertible.

Taking the 16 possible combinations of the elementary terms, and agreeing to cancel cosine terms whenever possible and limit the number of $\Gamma$ functions in the numerator or denominator to four, we obtain the following basic solutions,

$$\mathsf{A_1A_1B_1B_1} = \frac{\Gamma(s)\Gamma(s+1-c)}{\Gamma(s+1-a)\Gamma(s+1-b)} \qquad \mathsf{A_2A_1B_1B_1} = \frac{\Gamma(s+1-c)\Gamma(r+s)\Gamma(1-r-s)}{\pi\Gamma(1-s)\Gamma(s+1-a)\Gamma(s+1-b)}$$

$$\mathsf{A_1A_1B_1B_2} = \frac{\pi\Gamma(s)\Gamma(s+1-c)\Gamma(b-s)}{\Gamma(s+1-a)\Gamma(r+s)\Gamma(1-r-s)} \qquad \mathsf{A_2A_1B_1B_2} = \frac{\Gamma(s+1-c)\Gamma(b-s)}{\Gamma(1-s)\Gamma(s+1-a)}$$

$$\mathsf{A_1A_1B_2B_1} = \frac{\pi\Gamma(s)\Gamma(s+1-c)\Gamma(a-s)}{\Gamma(s+1-b)\Gamma(r+s)\Gamma(1-r-s)} \qquad \mathsf{A_2A_1B_2B_1} = \frac{\Gamma(s+1-c)\Gamma(a-s)}{\Gamma(1-s)\Gamma(s+1-b)}$$

$$\mathsf{A_1A_1B_2B_2} = \Gamma(s)\Gamma(s+1-c)\Gamma(a-s)\Gamma(b-s) \qquad \mathsf{A_2A_1B_2B_2} = \frac{\pi\Gamma(s+1-c)\Gamma(a-s)\Gamma(b-s)}{\Gamma(1-s)\Gamma(r+s)\Gamma(1-r-s)}$$

$$\mathsf{A_1A_2B_1B_1} = \frac{\Gamma(s)\Gamma(r+s)\Gamma(1-r-s)}{\pi\Gamma(s+1-a)\Gamma(s+1-b)\Gamma(c-s)} \qquad \mathsf{A_2A_2B_1B_1} = \frac{\Gamma(r+s)\Gamma(1-r-s)\Gamma(r'+s)\Gamma(1-r'-s)}{\pi^2\Gamma(1-s)\Gamma(c-s)\Gamma(s+1-a)\Gamma(s+1-b)}$$

$$\mathsf{A_1A_2B_1B_2} = \frac{\Gamma(s)\Gamma(b-s)}{\Gamma(s+1-a)\Gamma(c-s)} \qquad \mathsf{A_2A_2B_1B_2} = \frac{\Gamma(r+s)\Gamma(1-r-s)\Gamma(b-s)}{\pi\Gamma(1-s)\Gamma(c-s)\Gamma(s+1-a)}$$

$$\mathsf{A_1A_2B_2B_1} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(s+1-b)\Gamma(c-s)} \qquad \mathsf{A_2A_2B_2B_1} = \frac{\Gamma(r+s)\Gamma(1-r-s)\Gamma(a-s)}{\pi\Gamma(1-s)\Gamma(c-s)\Gamma(s+1-b)}$$

$$\mathsf{A_1A_2B_2B_2} = \frac{\pi\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)\Gamma(r+s)\Gamma(1-r-s)} \qquad \mathsf{A_2A_2B_2B_2} = \frac{\Gamma(a-s)\Gamma(b-s)}{\Gamma(1-s)\Gamma(c-s)}$$

where $r$ and $r'$ are arbitrary.

3.5. **Series solutions.** We now show how to determine solutions in series form using the inverse transform (3) and the method of contour integration.

3.6. **Integral solutions.**

3.7. **Inverse operators.** The following are inverse operators. If $a \geq 0$,

$$T = a + xD \qquad T^{-1} = x^{-a}\int_0^x t^{a-1}f(t)\,dt \quad \text{or} \quad x^{-a}\int_x^\infty t^{a-1}f(t)\,dt$$

depending on which integral exists.

## 4. Criteria for hypergeometric type

4.1. **Transforming by multiplicative factor.** Consider the following second order differential equaiton,

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0$$

Given Define the function $f(x)$ such that $y(x) = w(x)f(x)$.

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