Comments on *Section 6.1.3 Supercritical Andronov-Hopf* of Izhikevich (2007)

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1 Introduction

The book Izhikevich (2007) is excellent. However, when I reached *Section 6.1.3 Supercritical Andronov-Hopf* I did not feel comfortable with the presentation of a few formulas without derivation or explanation of where these formulas came from or what do they meant. Specifically, I could not understand the rationale

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for the change of variables in Equation 6.5, where the formula for parameter a in Equation 6.7 came from or what did it mean, how to derive the topological normal form in Equations 6.8 and 6.9, and how to compute parameter d in Equation 6.9.

Here I sketch the derivation of the topological normal form in Equations 6.8 and 6.9 and that of the formulas to calculate parameters a and d in these equations. In deriving the topological normal in Equations 6.8 and 6.9, I found that these equations do not represent a true topological normal form of a system undergoing an Andronov-Hopf bifurcation, but only an *approximation* of a true topological normal form. The formula for parameter a is given, without proof in Equation 6.7. On page 172 the author refers the reader to Exercise 6.17 to obtain the values of parameters a and d for an example $I_{N_{a,p}}+I_K$ model. However, this exercise gives neither the formula nor the value of parameter d. Furthermore, applying the formulas we derive below for parameters a and d to the example $I_{N_{a,p}} + I_K$ model on page 172 reveals a typo; the values of parameters a and d are interchanged in the penultimate paragraph on page 172, in the subsequent formula of the topological normal form of the $I_{N_{a,p}} + I_K$ model, and the center and right panels of Figure 6.12 were computed with interchanged parameters.

Section 2 provides a sketch of the derivation of the *approximate* topological normal form in Equations 6.8 and 6.9. Section 3 sketches the derivation of the formulas of parameters a and d of the topological normal form for a system undergoing an Andronov-Hopf bifurcation. Section 4 derives the formulas for $f(x, y)$ and $g(x, y)$ in Equation 6.6. Section 6 calculates the values of parameters a and d for the example on page 172. Finally, Section 7 presents evidence for the typo in the example on page 172.

2 Sketch of the derivation of the *approximate* topological normal form in Equations 6.8 and 6.9 of Izhikevich (2007)

Consider a planar system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \alpha),\ \mathbf{x} \in \mathbb{R}^2,\ \alpha \in \mathbb{R} \tag{1}$$

with a smooth function f, and an equilibrium at $\mathbf{x} = 0$ for parameter $\alpha = 0$. This system undergoes and Andronov-Hopf bifurcation iff its eigenvalues are $\lambda_{1,2} = \pm jw_0, w_0 > 0$. From Section 3.5 Generic Hopf Bifurcation of Kuznetsov (2004) a planar system undergoing an Andronov-Hopf bifurcation can be rewritten as

$$\dot{\mathbf{x}} = A(\alpha)\mathbf{x} + F(\mathbf{x}, \alpha) \tag{2}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in $\mathbf{x}$ starting with at least quadratic terms $F = \mathcal{O}(||\mathbf{x}||^2)$. Then, by

Lemma 3.3 in Kuznetsov (2004), the system in Equation 2 can be rewritten for sufficiently small $|\alpha|$ a single complex equation

$$\dot{z} = \lambda(\alpha)z + g(z, \bar{z}, \alpha), \tag{3}$$

where $g = \mathcal{O}(|z|^2)$ is a smooth function of $(z, \bar{z}, \alpha)$. Now from Lemma 3.6 (Poincaré normal form for the Hopf bifurcation) in Kuznetsov (2004) Equation 3 can be transformed, by an invertible parameter-dependent change of complex coordinate, for sufficiently small $|\alpha|$, into

$$\dot{w} = \lambda(\alpha)w + c_1(\alpha)w^2 w + \mathcal{O}(|w|^4). \tag{4}$$

Denoting $\lambda(\alpha) = c(\alpha) + jw(\alpha)$ and $c_1(\alpha) = a(\alpha) + jd(\alpha)$, by Problem 6.4 in Izhikevich (2007), the complex-valued Equation 4 can be written in polar coordinates as

$$\begin{aligned} \dot{r} &= c(\alpha)r + a(\alpha)r^3, \\ \dot{\varphi} &= w(\alpha) + d(\alpha)r^2 \end{aligned} \tag{5}$$

The system in Equation 5 is a true topological normal form of the sytem in Equation 1 undergoing an Andronov-Hopf bifurcation, because the former system can be obtained from the latter system by a change of variables that is an homeomorphism. However, the topological normal form in Equations 6.8 and 6.9 of Izhikevich (2007) is an approximation of the true toplogical normal form using $a(0)$ and $d(0)$ instead of $a(\alpha)$ and $d(\alpha)$ in Equation 5.

To obtain exact values of a and d independent of any parameter, one could use a topological normal form as in Equation 4 but with a coefficient of the cubic term independent of any parameter (e.g., $c_1(\alpha) = c_1$ in Equation 4). This new topological normal form is given in Lemma 3.7 of Kuznetsov (2004). In this lemma Kuznetsov (2004) attains a topological normal form where the coefficient of the cubic term is $\text{sign}(l_1(\beta))$. Because $l_1(\beta)$ is a continuous function of β, with $l_1(0) \neq 0$, for small values of β the sign of $l_1(\beta)$ is constant (i.e., $\text{sign}(l_1(\beta)) = s$). Therefore, by replacing $\text{sign}(l_1(\beta))$ with s in the previous topological normal form, for small values of β, Kuznetsov attains a new topological normal form with the coefficient of the cubic term independent of any parameter.

3 Sketch of derivation of formulas for parameters a and d in Equations 6.8 and 6.9 of Izhikevich (2007)

As shown by the end of Section 2 parameters a and d in Equations 6.8 and 6.9 of Izhikevich (2007) are $a = Re\{c_1(0)\}$ and $d = Im\{c_1(0)\}$, with $c_1(0)$ given in

Equation 4. To calculate $c_1(0)$ we first apply the change of variables in Equation 6.5 of Izhikevich (2007) to convert the system undergoing an Andronov-Hopf bifurcation in Equation 6.4 of Izhikevich (2007), with parameter $b = 0$, into the form in Equation 6.6 of Izhikevich (2007). Then we solve Problem (3.16) in Kuznetsov (2004) to obtain $c_1(0)$ from the system in Equation 6.6 of Izhikevich (2007).

Section 3.1 provides the rationale for Equations 6.5 and 6.6 of Izhikevich (2007). Section 3.2 sketches the solution of Problem 3.16 in Kuznetsov (2004).

3.1 Rationale for Equations 6.5 and 6.6 in Izhikevich (2007)

The rationale for the change of variables in Equation 6.5 of Izhikevich (2007) is given in Lemma 3.3 of Kuznetsov (2004). This lemma proves that a system of differential equations

$$\dot{\mathbf{x}} = A(\alpha)\mathbf{x} + F(\mathbf{x}, \alpha)$$

where $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ is a smooth vector function whose components have Taylor expansions in $\mathbf{x}$ starting with at least quadratic terms, can be written as a single differential equation on a complex variable z:

$$\dot{z} = \lambda(\alpha)z + g(z, \bar{z}, \alpha) : \tag{6}$$

with $\lambda(\alpha)$ being the eigenvalue of $A(\alpha)$ corresponding to eigenvector $\mathbf{q}(\alpha)$ (i.e., $A(\alpha)\mathbf{q}(\alpha) = \lambda(\alpha)\mathbf{q}(\alpha)$) and z being the coordinate associated with $\mathbf{q}(\alpha)$ in the expansion of $\mathbf{x}$ in the base $\{\mathbf{q}(\alpha), \bar{\mathbf{q}}(\alpha)\}$,

$$\mathbf{x} = z\mathbf{q}(\alpha) + \bar{z}\bar{\mathbf{q}}(\alpha), \tag{7}$$

and

$$g(z, \bar{z}, \alpha) = \langle \mathbf{p}(\alpha), F(z\mathbf{q}(\alpha) + \bar{z}\bar{\mathbf{q}}(\alpha), \alpha) \rangle \tag{8}$$

with $\{\mathbf{p}(\alpha), \bar{\mathbf{p}}(\alpha)\}$ a base of eigenvectors of $A(\alpha)^T$ biorthogonal with the base $\{\mathbf{q}(\alpha), \bar{\mathbf{q}}(\alpha)\}$ of $A(\alpha)$. The Taylor series expansion of the right-hand-side of the differential equations in Equation 6.4 of Izhikevich (2007), for $b = 0$ can be written as:

$$\begin{bmatrix} \dot{v} \\ \dot{u} \end{bmatrix} = A(0) \begin{bmatrix} v \\ u \end{bmatrix} + \begin{bmatrix} \tilde{F}(v, u, 0) \\ \tilde{G}(v, u, 0) \end{bmatrix} \tag{9}$$

with $A(0)$ being the jacobian matrix of the right hand side of the differential equations in Equation 6.4 of Izhikevich (2007)

$$A(0) = \begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix} \tag{10}$$

where F_u, F_v, G_u, G_v are partial derivatives evaluated at $(u, v, b) = (0, 0, 0)$. Equation 9 holds because it is assummed that $(v, u) = (0, 0)$ is an equilibrium for $b = 0$ of the system of differential equations in Equation 6.4 of Izhikevich (2007), then $F(0, 0, 0) = G(0, 0, 0) = 0$, and the Taylor series expansion of the right-hand-side of Equation 6.4 in Izhikevich (2007) has no constant term.

Now, because it is assummed that the system in Equation 6.4 in Izhikevich (2007) undergoes an Andronov-Hopf bifurcation at the equlibrium $(v, u) = (0, 0)$, by the non-hyperbolicity condition of the Andronov-Hopf bifurcation, the eigenvalues of $A(0)$ in Equation 9 are $\pm jw$. Then by Equation 6:

$$\dot{z} = jwz + g(z, \bar{z}, 0) \tag{11}$$

with z being the coordinate associated with $\mathbf{q}(0)$ in the expansion of $[v, u]^T$ in the base $\{\mathbf{q}(0), \bar{\mathbf{q}}(0)\}$. That is, from Equation 7,

$$\begin{bmatrix} v \\ u \end{bmatrix} = z\mathbf{q}(0) + \bar{z}\bar{\mathbf{q}}(0) \tag{12}$$

Separating the real and imaginary parts of z in Equation 11 we obtain:

$$\begin{aligned} Re\{\dot{z}\} &= -wIm\{z\} + Re\{g(z, \bar{z}, 0)\} \\ Im\{\dot{z}\} &= wRe\{z\} + Im\{g(z, \bar{z}, 0)\} \end{aligned}$$

and calling

$$x = Re\{z\}, \tag{13}$$
$$y = Im\{z\}, \tag{14}$$
$$f(x, y) = Re\{g(z, \bar{z}, 0)\}, \tag{15}$$
$$g(x, y) = Im\{g(z, \bar{z}, 0)\}, \tag{16}$$

we obtain

$$\begin{aligned} \dot{x} &= -wy + f(x, y) \\ \dot{y} &= wx + g(x, y) \end{aligned}$$

as in Equation 6.6 in Izhikevich (2007).

To express x and y as a function of v and u, as in Equation 6.5 of Izhikevich (2007), from Equations 13 and 14, we want to express z as a function of v and

u. One can build the biorthogonal base $\{\mathbf{p}(0), \bar{\mathbf{p}}(0)\}$ corresponding to the base of eigenvectors of $A(0)$, $\{\mathbf{q}(0), \bar{\mathbf{q}}(0)\}$ and, from Equation 12, compute z as

$$z = \langle \mathbf{p}(0), \begin{bmatrix} v \\ u \end{bmatrix} \rangle \tag{17}$$

Because $\bar{\mathbf{q}}(0)$ is the eigenvector of $A(0)$ (Equation 10) with eigenvalue $-jw$, we have

$$\begin{bmatrix} F_v + jw & F_u \\ G_v & G_u + jw \end{bmatrix} \begin{bmatrix} \bar{q}_1(0) \\ \bar{q}_2(0) \end{bmatrix} = 0$$

Then

$$(F_v + jw)\bar{q}_1(0) + F_u \bar{q}_2(0) = 0 \tag{18}$$

Due to the biorthogonality of the bases $\{\mathbf{p}(0), \bar{\mathbf{p}}(0)\}$ and $\{\mathbf{q}(0), \bar{\mathbf{q}}(0)\}$ we have $\langle \mathbf{p}(0), \bar{\mathbf{q}}(0) \rangle = 0$. Thus, from Equation 18, we obtain

$$\mathbf{p}(0) = k \begin{bmatrix} F_v - jw \\ F_u \end{bmatrix} \tag{19}$$

where k is a complex constant. Now, from Equations 17 and 19

$$\begin{aligned} z =& \langle \mathbf{p}(0), \begin{bmatrix} v \\ u \end{bmatrix} \rangle \\ =& \bar{k} v (F_v + jw) + \bar{k} u F_u \\ =& \bar{k}(uF_u + vF_v) + j\bar{k}wv \end{aligned} \tag{20}$$

By using $k = j/w$ in Equation 20 we obtain $z = v - j(uF_u + vF_v)/w$. Then $x = Re\{z\} = v$, and $y = Im\{z\} = -(uF_u + vF_v)/w = -(uF_u + xF_v)/w$, or $v = x$ and $F_u u = -F_v x - wy$ as in Equation 5.5 of Izhikevich (2007).

3.2 Solution to Problem 3.16 in Kuznetsov (2004)

Given a system as that in Equation 6.6 of Izhikevich (2007), Problem 3.16 in Kuznetsov (2004) asks to compute $c_1(0)$. The solution is $c_1(0) = a(0) + jd(0) = a + jd$ with

$$\begin{aligned} a =& \frac{1}{16}((f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) + \\ & \frac{1}{w}(g_{yy}f_{yy} - g_{xx}f_{xx} + f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}))), \end{aligned} \tag{21}$$

$$
\begin{aligned}
d =& \frac{1}{16}((g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy}) + \\
& \frac{1}{3w}((f_{xy}g_{xx} + g_{xy}f_{yy}) - 2(g_{yy}^2 + f_{xx}^2) - 2(f_{xy}^2 + g_{xy}^2) - \\
& \quad 5(g_{xx}^2 + f_{yy}^2) + 5(f_{xy}g_{yy} + g_{xy}f_{xx}) - \\
& \quad 5(g_{xx}g_{yy} + f_{xx}f_{yy}))).
\end{aligned} \tag{22}
$$

To derive this solution, we first rewrite Equation 6.6 of Izhikevich (2007) in matrix form as

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A(0) \begin{bmatrix} x \\ y \end{bmatrix} + F\left(\begin{bmatrix} x \\ y \end{bmatrix}, 0 \right) \tag{23}
$$

with

$$
A_0 = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}, \tag{24}
$$

$$
F\left(\begin{bmatrix} x \\ y \end{bmatrix}, 0 \right) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}, \tag{25}
$$

and f and g having Taylor series expansions starting with at least quadratic terms

$$
\begin{aligned}
f(x, y) =& \frac{1}{2}f_{xx}x^2 + f_{xy}xy + \frac{1}{2}f_{yy} + \\
& \frac{1}{6}f_{xxx}x^3 + \frac{1}{2}f_{xxy}x^2y + \frac{1}{2}f_{xyy}xy^2 + \frac{1}{6}f_{yyy}y^3 + \dots,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
g(x, y) =& \frac{1}{2}g_{xx}x^2 + g_{xy}xy + \frac{1}{2}g_{yy} + \\
& \frac{1}{6}g_{xxx}x^3 + \frac{1}{2}g_{xxy}x^2y + \frac{1}{2}g_{xyy}xy^2 + \frac{1}{6}g_{yyy}y^3 + \dots
\end{aligned} \tag{27}
$$

with all partial derivatives of f and g evaluated at the origin. From Equation 3.18 of Kuznetsov (2004),

$$
c_1(0) = \frac{j}{2w}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \tag{28}
$$

where, from p. 91 of Kuznetsov (2004), g_{kl} are the coefficients of the Taylor series expansion of the function $g(z, \bar{z}, 0)$ in Equation 8,

$$
g(z, \bar{z}, 0) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(0) z^k \bar{z}^l. \tag{29}
$$

To calculate $c_1(0)$ we will expand $g(z, \bar{z}, 0)$ in Taylor and extract the coefficients g_{11}, g_{20}, g_{02}, and g_{21} needed in Equation 28. We choose $\mathbf{p}(0) = [-j, -1]^T$ and $\mathbf{q}(0) = [-j/2, -1/2]^T$. Then, from Equation 8,

$$g(z, \bar{z}, 0) = jf(z\mathbf{q}(0) + \bar{z}\bar{\mathbf{q}}(0)) - g(z\mathbf{q}(0) + \bar{z}\bar{\mathbf{q}}(0)). \tag{30}$$

Inserting

$$z\mathbf{q}(0) + \bar{z}\bar{\mathbf{q}}(0) = -\frac{1}{2}\left[\begin{array}{c} jz - j\bar{z}, \\ z + \bar{z} \end{array}\right] = \left[\begin{array}{c} x \\ y \end{array}\right]$$

into Equations 26 and 27 we obtain

$$\begin{aligned}
f(z\mathbf{q}(0) + \bar{z}\bar{\mathbf{q}}(0)) =& (-\frac{1}{8}f_{xx} + \frac{1}{8}f_{yy} + j\frac{1}{4}f_{xy})z^2 + \\
& (\frac{1}{4}f_{xx} + \frac{1}{4}f_{yy})z\bar{z} + \\
& (-\frac{1}{8}f_{xx} + \frac{1}{8}f_{yy} - j\frac{1}{4}f_{xy})\bar{z}^2 + \\
& (\frac{1}{16}f_{xxy} - \frac{1}{48}f_{yyy} + j(\frac{1}{48}f_{xxx} - \frac{1}{16}f_{xyy}))z^3 + \\
& (-\frac{1}{16}f_{xxy} - \frac{1}{16}f_{yyy} - j(\frac{1}{16}f_{xxx} + \frac{1}{16}f_{xyy}))z^2\bar{z} + \\
& (-\frac{1}{16}f_{xxy} - \frac{1}{16}f_{yyy} - j(\frac{1}{16}f_{xxx} + \frac{1}{16}f_{xyy}))z\bar{z}^2 + \\
& (\frac{1}{16}f_{xxy} - \frac{1}{48}f_{yyy} - j(\frac{1}{48}f_{xxx} - \frac{1}{16}f_{xyy}))\bar{z}^3
\end{aligned} \tag{31}$$

$$\begin{aligned}
g(z\mathbf{q}(0) + \bar{z}\bar{\mathbf{q}}(0)) =& (-\frac{1}{8}g_{xx} + \frac{1}{8}g_{yy} + j\frac{1}{4}g_{xy})z^2 + \\
& (\frac{1}{4}g_{xx} + \frac{1}{4}g_{yy})z\bar{z} + \\
& (-\frac{1}{8}g_{xx} + \frac{1}{8}g_{yy} - j\frac{1}{4}g_{xy})\bar{z}^2 + \\
& (\frac{1}{16}g_{xxy} - \frac{1}{48}g_{yyy} + j(\frac{1}{48}g_{xxx} - \frac{1}{16}g_{xyy}))z^3 + \\
& (-\frac{1}{16}g_{xxy} - \frac{1}{16}g_{yyy} - j(\frac{1}{16}g_{xxx} + \frac{1}{16}g_{xyy}))z^2\bar{z} + \\
& (-\frac{1}{16}g_{xxy} - \frac{1}{16}g_{yyy} - j(\frac{1}{16}g_{xxx} + \frac{1}{16}g_{xyy}))z\bar{z}^2 + \\
& (\frac{1}{16}g_{xxy} - \frac{1}{48}g_{yyy} - j(\frac{1}{48}g_{xxx} - \frac{1}{16}g_{xyy}))\bar{z}^3
\end{aligned} \tag{32}$$

Using Equations 31 and 32 in Equation 16 we obtain the Taylor series expansion of $g(z, \bar{z}, 0)$, and from this expansion we extract the coefficients g_{11}, g_{20}, g_{02}, and g_{21} needed in Equation 28. The obtained coefficients are

$$
\begin{aligned}
g_{20} = g_{z^2} =& \frac{1}{4}((g_{xx} - g_{yy} - 2f_{xy}) + j(f_{yy} - f_{xx} - 2g_{xy})) && (33)\\
g_{11} = g_{z\bar{z}} =& \frac{1}{4}((-g_{xx} - g_{yy}) + j(f_{yy} + f_{xx})) && (34)\\
g_{02} = g_{\bar{z}^2} =& \frac{1}{4}((g_{xx} - g_{yy} + 2f_{xy}) + j(f_{yy} - f_{xx} + 2g_{xy})) && (35)\\
g_{21} = g_{z^2\bar{z}} =& \frac{1}{8}((f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) + \\
& j(-f_{xxy} + g_{xxx} - f_{yyy} + g_{xyy})) && (36)
\end{aligned}
$$

From Equation 16 and Equations 33-36 we obtain

$$
\begin{aligned}
a =& Re\{c_1(0)\} = -\frac{1}{2w} Im\{g_{20}g_{11}\} + \frac{1}{2} Re\{g_{21}\} = \\
& \frac{1}{16}((f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) + \\
& \quad \frac{1}{w}(g_{yy}f_{yy} - g_{xx}f_{xx} + f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}))), \\
d =& Im\{c_1(0)\} = \frac{1}{2w}(Re\{g_{20}g_{11}\} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{1}{2} Im\{g_{21}\} = \\
& \frac{1}{16}((g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy}) + \\
& \quad \frac{1}{3w}((f_{xy}g_{xx} + g_{xy}f_{yy}) - 2(g_{yy}^2 + f_{xx}^2) - 2(f_{xy}^2 + g_{xy}^2) - \\
& \qquad 5(g_{xx}^2 + f_{yy}^2) + 5(f_{xy}g_{yy} + g_{xy}f_{xx}) - \\
& \qquad 5(g_{xx}g_{yy} + f_{xx}f_{yy}))),
\end{aligned}
$$

as in Equations 21 and 22.

4 Derivation of formulas for $f(x, y)$ and $g(x, y)$ in Equation 6.6 of Izhikevich (2007)

Equations 15 and 16 give formulas for $f(x, y)$ and $g(x, y)$ in Equation 6.6 of Izhikevich (2007). However, to use these formulas one first needs to compute the function $g(z, \bar{z}, 0)$. Below we derive formulas for $f(x, y)$ and $g(x, y)$ as a function of the functions $F(v, u)$ and $G(v, u)$ in Equation 6.4 of Izhikevich (2007), that do not require function $g(z, \bar{z}, 0)$.

From the change of variables in Equation 6.5 of Izhikevich (2007) we have

$$
\begin{aligned}
\dot{x} = \dot{v} = F(v, u, 0) = F(x, -\frac{1}{F_n}(F_v x + wy)) \\
= -wy + F(x, -\frac{1}{F_n}(F_v x + wy), 0) + wy = -wy + f(x, y),
\end{aligned}
$$

with

$$f(x,y) = F(x, -\frac{1}{F_n}(F_v x + wy), 0) + wy. \tag{37}$$

Also, from the change of variables in Equation 6.5 of Izhikevich (2007) we obtain, for $w \neq 0$,

$$y = -\frac{1}{w}(F_u u + F_v x).$$

Then

$$\begin{aligned}
\dot{y} &= -\frac{1}{w}(F_u \dot{u} + F_v \dot{x}) \\
&= -\frac{1}{w}(F_u G(v,u,0) + F_v F(v,u,0)) \\
&= -\frac{1}{w}(F_u G(x, -\frac{1}{F_n}(F_v x + wy), 0) + F_v F(x, -\frac{1}{F_n}(F_v x + wy), 0)) \\
&= wx - \frac{1}{w}(F_u G(x, -\frac{1}{F_n}(F_v x + wy), 0) + F_v F(x, -\frac{1}{F_n}(F_v x + wy), 0)) - wx \\
&= wx + g(x,y),
\end{aligned}$$

with

$$\begin{aligned}
g(x,y) = &-\frac{1}{w}(F_u G(x, -\frac{1}{F_n}(F_v x + wy), 0) + F_v F(x, -\frac{1}{F_n}(F_v x + wy), 0)) \\
&- wx.
\end{aligned} \tag{38}$$

5 Partial derivatives of $f(x,y)$ and $g(x,y)$ for the calculation of parameters a and d in Equations 6.8 and 6.9 in Izhikevich (2007)

Although tedious, it is not difficult to compute the partial derivatives of $f(x,y)$ and $g(x,y)$ in Equation 37 and 38, respectively, and evaluate them at $x = 0$ and $y = 0$. We obtained:

$$\begin{aligned}
f_{xx} =& F_{VV} + 2F_{Vn}\left(-\frac{F_V}{F_n}\right) + F_{nn}\left(-\frac{F_V}{F_n}\right)^2 \\
f_{xxx} =& F_{VVV} + 3F_{VVn}\left(-\frac{F_V}{F_n}\right) + 3F_{Vnn}\left(-\frac{F_V}{F_n}\right)^2 + F_{nnn}\left(-\frac{F_V}{F_n}\right)^3
\end{aligned}$$

$$
\begin{aligned}
f_{xxy} =& \left(-\frac{w}{F_n}\right)\left(F_{VVn} + 2F_{Vnn}\left(-\frac{F_V}{F_n}\right) + F_{nnn}\left(-\frac{F_V}{F_n}\right)^2\right) \\
f_{yy} =& F_{nn}\left(-\frac{w}{F_n}\right)^2 \\
f_{yyy} =& F_{nnn}\left(-\frac{w}{F_n}\right)^3 \\
f_{xyy} =& \left(-\frac{w}{F_n}\right)^2\left(F_{Vnn} + F_{nnn}\left(-\frac{F_V}{F_n}\right)\right) \\
f_{xy} =& \left(-\frac{w}{F_n}\right)\left(F_{Vn} + F_{nn}\left(-\frac{F_V}{F_n}\right)\right) \\
g_{xx} =& -\frac{1}{w}\left(F_V\left(F_{VV} + 2F_{Vn}\left(-\frac{F_V}{F_n}\right) + F_{nn}\left(-\frac{F_V}{F_n}\right)^2\right) + \right. \\
& \left. F_n\left(G_{VV} + 2G_{Vn}\left(-\frac{F_V}{F_n}\right) + G_{nn}\left(-\frac{F_V}{F_n}\right)^2\right)\right) \\
g_{xxx} =& -\frac{1}{w}\left(F_V\left(F_{VVV} + 3F_{VVn}\left(-\frac{F_V}{F_n}\right) + 3F_{Vnn}\left(-\frac{F_V}{F_n}\right)^2 + F_{nnn}\left(-\frac{F_V}{F_n}\right)^3\right) + \right. \\
& \left. F_n\left(G_{VVV} + 3G_{VVn}\left(-\frac{F_V}{F_n}\right) + 3G_{Vnn}\left(-\frac{F_V}{F_n}\right)^2 + G_{nnn}\left(-\frac{F_V}{F_n}\right)^3\right)\right) \\
g_{xxy} =& \frac{1}{F_n}\left(F_V\left(F_{VVn} + 2F_{Vnn}\left(-\frac{F_V}{F_n}\right) + F_{nnn}\left(-\frac{F_V}{F_n}\right)^2\right) + \right. \\
& \left. F_n\left(G_{VVn} + 2G_{Vnn}\left(-\frac{F_V}{F_n}\right) + G_{nnn}\left(-\frac{F_V}{F_n}\right)^2\right)\right) \\
g_{yy} =& -\frac{w}{F_n^2}(F_V F_{nn} + F_n G_{nn}) \\
g_{yyy} =& \frac{w^2}{F_n^3}(F_V F_{nnn} + F_n G_{nnn}) \\
g_{xyy} =& -\frac{w}{F_n^2}(F_V(F_{Vnn} + F_{nnn}) + F_n(G_{Vnn} + G_{nnn})) \\
g_{xy} =& \frac{1}{F_n}\left(F_V\left(F_{Vn} + F_{nn}\left(-\frac{F_V}{F_n}\right)\right) + F_n\left(G_{Vn} + G_{nn}\left(-\frac{F_V}{F_n}\right)\right)\right)
\end{aligned}
$$

6 Parameters a and d in the example on page 172 of Izhikevich (2007)

The example on page 172 of Izhikevich (2007) calculates the *approximate* topological normal form for the $I_{Na_p} + I_K$ model

$$
\begin{aligned}
\dot{V} = \frac{1}{C}((I + I_0) - g_L(V + V_0 - E_L) \\
- g_{Na} m_\infty(V + V_0)(V + V_0 - E_{Na}) \\
- g_K(n + n_0)(V + V_0 - E_K)) = F(V, n, I)
\end{aligned} \tag{39}
$$

$$
\dot{n} = \frac{n_\infty(V + V_0) - (n + n_0)}{\tau} = G(V, n, I) \tag{40}
$$

where the V, n, and I have been centered at the Andronov-Hopf bifurcation point $V_0 = -56.4815$, $n_0 = 0.0914$, and $I_0 = 14.659$, respectievly. To calculate the value of the parameters a and d for the topological normal form in Equations 6.8 and (6.9), we first compute the partial derivaties of $F(V, n, I)$ and $G(V, n, I)$ in Equations 39 and 40, respectively, evaluated at $V = 0$, $n = 0$, $I = 0$

$$
\begin{aligned}
F_V =& \frac{1}{C}(-g_L - g_{Na}(m'_\infty(V_0)(V_0 - E_{Na}) + m_\infty(V_0)) - g_K n_0) \\
F_{VV} =& -\frac{g_{Na}}{C}(m''_\infty(V_0)(V_0 - E_{Na}) + m'_\infty(V_0)) \\
F_{VVV} =& -\frac{g_{Na}}{C}(m'''_\infty(V_0)(V_0 - E_{Na}) + 2m''_\infty(V_0)) \\
F_n =& \frac{1}{C}(-g_K(V_0 - E_K)) \\
F_{nn} =& 0 \\
F_{nnn} =& 0 \\
F_{Vn} =& -\frac{g_K}{C} \\
F_{VVn} =& 0 \\
F_{Vnn} =& 0 \\
G_V =& \frac{n'_\infty(V_0)}{\tau} \\
G_{VV} =& \frac{n''_\infty(V_0)}{\tau} \\
G_{VVV} =& \frac{n'''_\infty(V_0)}{\tau} \\
G_n =& -\frac{1}{\tau} \\
G_{nn} =& 0 \\
G_{nnn} =& 0 \\
G_{Vn} =& 0 \\
G_{VVn} =& 0 \\
G_{Vnn} =& 0
\end{aligned}
$$

Next, we use these partial derivatives to obtain the partial derivatives in the previous section. Finally, we use the partial derivaties in the previous section to obtain the values of the paramters a and d from Equations 21 and 22, respectively. This gives $a = -0.002970$ and $d = -0.002613$. Code for the calculation of the values of parameters a and d is given in:
`https://sccn.ucsd.edu/~rapela/dynamicalSystems/izhikevich07/ch6/code/`

7 Evidence for typo in the example on page 172 of Izhikevich (2007)

In the example on page 172, Izhikevich (2007) reports parameters values $a = -0.0026$ and $d = -0.0029$ for the topological normal form in Equations 6.8 and (6.9) of the $I_{Na,p} + I_K$ model. These parameters values are different from the ones derived in the previous section ($a = -0.002970$ and $d = -0.002613$). That the parameter value for a (d) reported in Izhikevich (2007) is very similar to the parameter value for d (a) derived in the previous section suggests a typo in Izhikevich (2007), where the values of parameters a and d have been interchanged. To check which set of parameters values is more adecuate, the ones derived in the previous section or the ones provided in Izhikevich (2007), we compared descriptors (amplitude and frequency) of limit cycles in simulations of the $I_{Na,p} + I_K$ model of the example on page 172 of Izhikevich (2007) with theoretical values of these descriptors computed from the parmeters derived in the previous section and from the parameters given in Izhikevich (2007). We computed theoretical values using formulas on page 173 of Izhikevich (2007). Figures 1a and 1b are as the center and left panels, respectively, in Figure 6.12 in Izhikevich (2007). They show that the parameters values derived in the previous section yield theoretical values of the amplitude and frequency of limit cycles closer to descriptors extracted from model simulations than the parameter values provided in Izhikevich (2007). This indicates that the parameter values derived in the previous section are more adequate for the toplogical normal form in Equations 6.8 and 6.9 of the $I_{Na,p} + I_K$ model of the example on page 172 of Izhikevich (2007) than later parameter values reported in Izhikevich (2007), specially for input currents closer to the bifurcation point, $I_{ah} = 14.659$.

We simulated the $I_{Na,p} + I_K$ model in the example of page 172 of Izhikevich (2007) and the topological normal form in Equations 6.8 and 6.9 with the parameters derived in the previous section and with the parameters in Izhikevich (2007). For these simulations we used a constant injected dc-current $I = 17$. The blue trace in Figure 2 plots voltages from the $I_{Na,p} + I_K$ model. The green and red traces plot voltages from the topological normal forms with the parameters derived in the previous section and with the parameters in Izhikevich (2007), respectively. We see that the voltages from the topological normal form with the parameters derived in the previous section better approximate the voltages

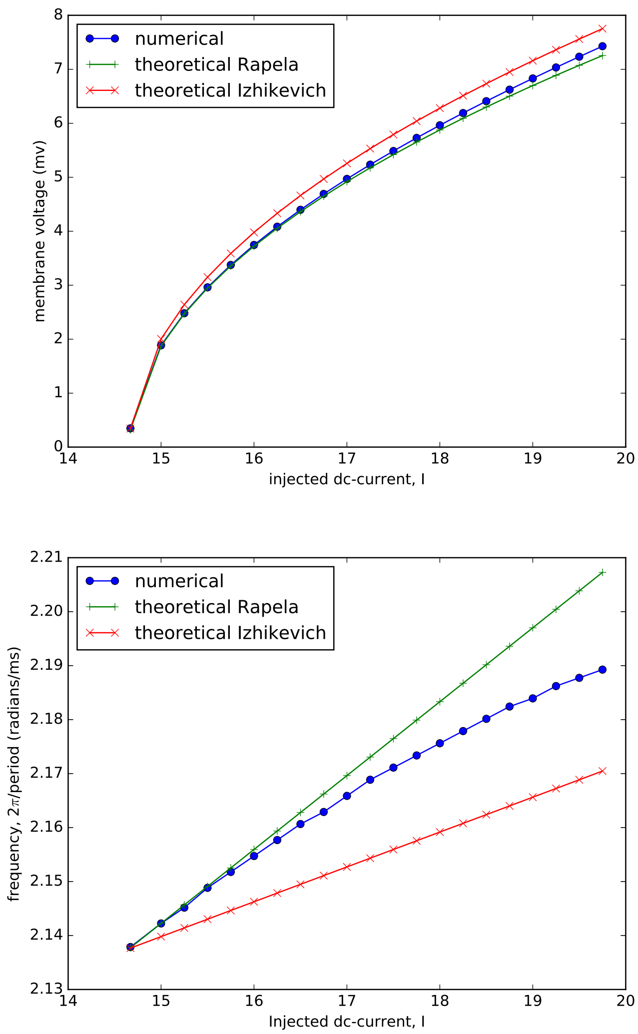

Figure 1: Descriptors (amplitude in panel (a) and frequency in panel (b)) of limit cycles next to an Andronov-Hopf bifurcation for the $I_{Na,p} + I_K$ model on the example of page 172 of Izhikevich (2007) as a function of the input current to the model. Blue traces give these descriptors values derived from numerical simulations. Green traces provide theoretical values of these descriptors, from the formulas given on page 173 of Izhikevich (2007), using parameters $a = -0.00297$ and $d = -0.002613$ derived in the previous section. Red traces are as the green traces but for the parameters $a = 0.0026$ and $d = 0.0029$ used in the example of page 172 of Izhikevich (2007). Theoretical values obtained using the parameters a and d from the previous section better approximate descriptors derived from numerical simulations than theoretical values obtained using the parameters in Izhikevich (2007).

from the model than the voltages from the topological normal form with the parameters in Izhikevich (2007). Thus, Figure 2 again shows that the parameters a and d of the topological normal form in Equations 6.8 and 6.9 derived in the previous section are more adequate than the ones derived in Izhikevich (2007).

References

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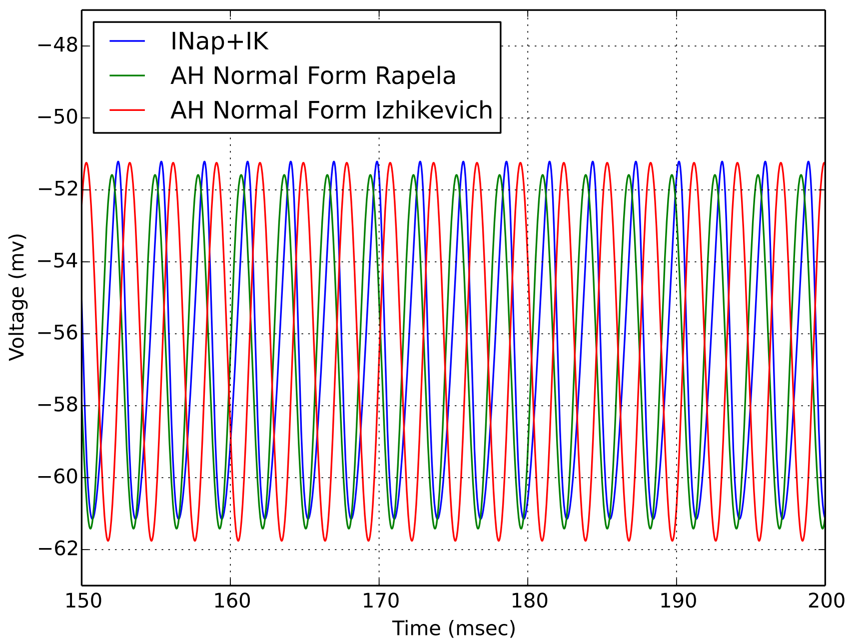

Figure 2: Voltages from the simulation close to an Andronov-Hopf bifurcation (input current I=17) of the $I_{Na,p} + I_K$ model on the example of page 172 of Izhikevich (2007) and from its topological normal form given in Equations 6.8 and 6.9 of Izhikevich (2007). The blue trace gives voltages from the simulation of the $I_{Na,p} + I_K$ model. The green trace provides voltages from the simulation of the topological normal form using the parameters $a = -0.00297$ and $d = -0.002613$ derived in the previous section. The red trace is as the green one, but for the parameters $a = 0.0026$ and $d = 0.0029$ used in the example on page 172 of Izhikevich (2007). Voltages from the topological normal form with the parameters from the previous section are better approximations to the voltages from the simulation of the $I_{Na,p} + I_K$ model than voltages from the topological normal form with the parameters used in the example on page 172 of Izhikevich (2007).