One-Dimensional Approximation of a Multi-Dimensional Neuronal System

Joaquin Rapela*

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1 Theory

The dynamics of a **multi-dimensional** dynamical system near a saddle-node bifurcation can be reduced to that of the **one-dimensional** topological normal form (Izhikevich, 2007, p.163):

$$\dot{V} = c(b - b_{sn}) + a(V - V_{sn})^2 \tag{1}$$

where

$$a = \frac{1}{2} \frac{\partial^2 \mathbf{I}(V, b_{sn})}{\partial V^2} \bigg|_{V=V_{sn}} \tag{2}$$

$$c = \frac{\partial \mathbf{I}(V_{sn}, b)}{\partial b} \bigg|_{b=b_{sn}} \tag{3}$$

b represents the bifurcation parameter, and (b_{sn}, V_{sn}) is the bifurcation point.

Note: by *near a saddle-node bifurcation* we mean that the trajectory of the neuronal system should pass close to a saddle-node bifurcation point.

2 Example INap+IK model

An INap+IK model with high-threshold parameters undergoes a saddle-node bifurcation at $V_{sn} = -60.9325$ and $n_{sn} = 0.0007$, with $I_{sn} = 4.51$ (Izhikevich, 2007, Example: The INap+IK-model on page 163). With these parameters we obtain $a = 0.1887$ and $c = 1$ from Eqs. 2 and 3. Thus, the equivalent one-dimensional model takes the form:

*rapela@ucsd.edu

$$\dot{V} = (I - 4.51) + 0.1887(V + 60.9325)^2 \tag{4}$$

which can be solved analytically. For $I > I_{sn}$ the solution is:

$$V(t) = V_{sn} + \sqrt{\frac{c}{a}(I - I_{sn})}\tan(a\,c\,(I - I_{sn})\,t + c_0) \tag{5}$$

If we reset the voltage to $V(t + \Delta t) = V_{reset}$ when $V(t) > V_{thr}$, we obtain the Quadratic Integrate and Fire (QIF) model (Izhikevich, 2007, p. 80).

For different values of the input current, Figures 1-4 plot the phase space of the INap+IK model and its voltage traces along with the voltage traces of the QIF model [1].

The phase spaces in the top panels of Figures 1-4 show that for the chosen parameters and input current values the INap+IK model is close to the saddle-node bifurcation. Therefore, the QIF should be a good approximation, as demonstrated by the voltage traces in the bottom panels of Figures 1-4. However, if we change the time constant of the n gating variable of the INap+IK model from 1.0 to 0.152 and use an input current $I = 5.0$, the trajectory INap+IK model moves away from the saddle-node bifurcation point and its approximation by the QIF model becomes poor, as illustrated in Figure 5.

References

Izhikevich, E. (2007). *Dynamical systems in neuroscience.* MIT press.

[1] We have added a constant delay of 5.1 ms at times when the voltage reaches threshold (i.e., $V(t) = V_{thr}$) to account for the approximately constant spike generation time in the INap+IK model.

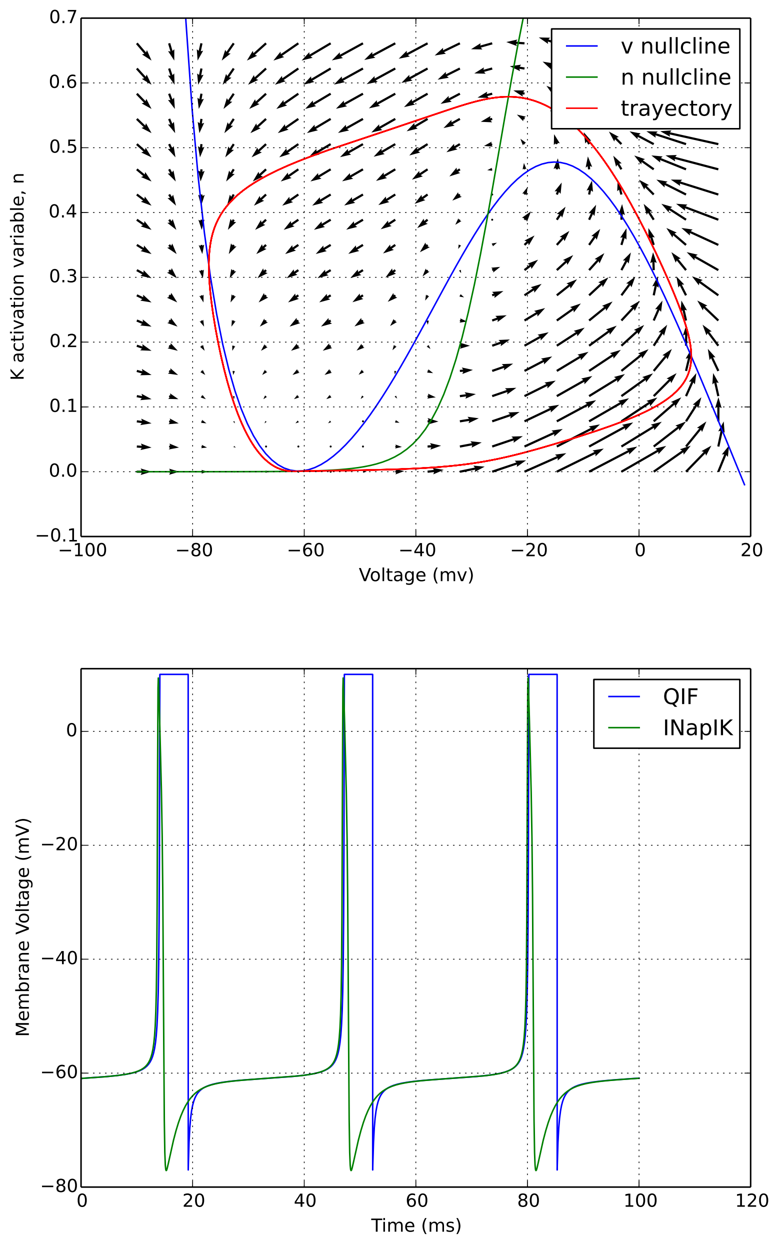


Figure 1: Top: phase space of the INap+IK model with high-threshold parameters. Bottom: voltage trace of the INap+IK model (green trace), and its approximation by a QIF model (blue trace). Input current 4.58 mA.

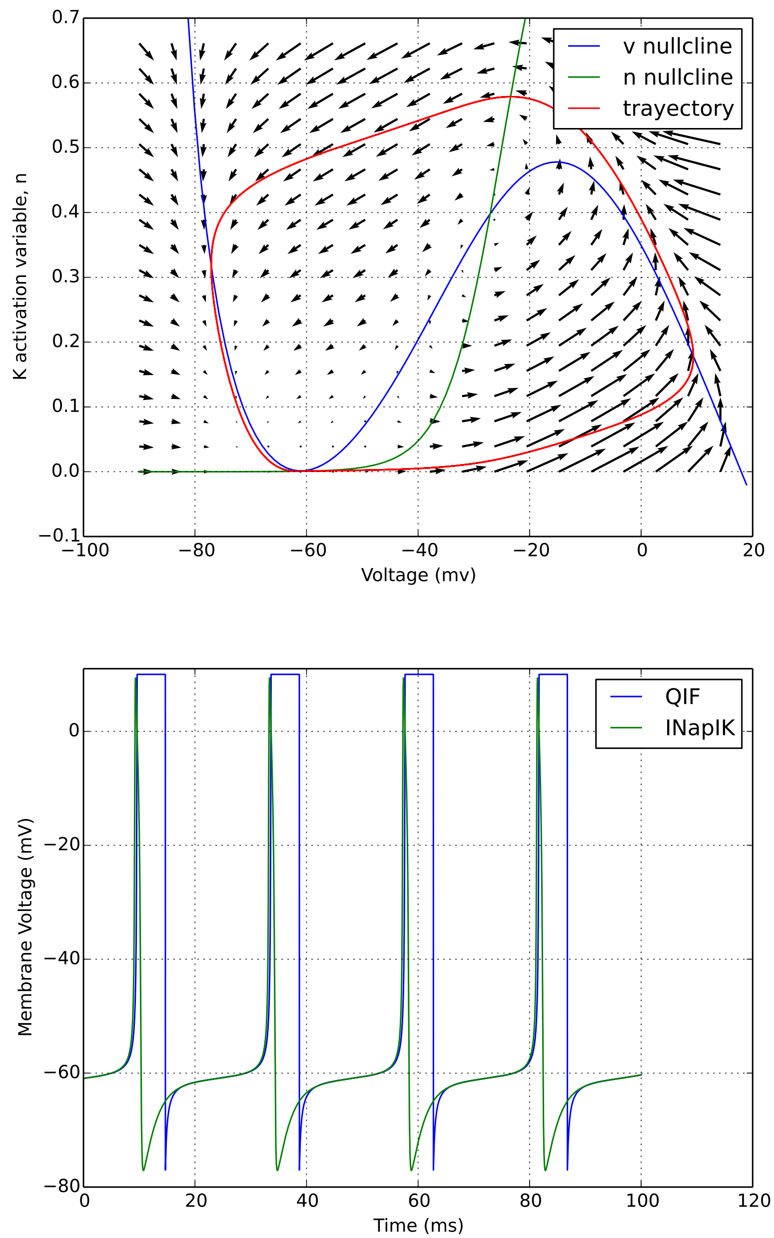
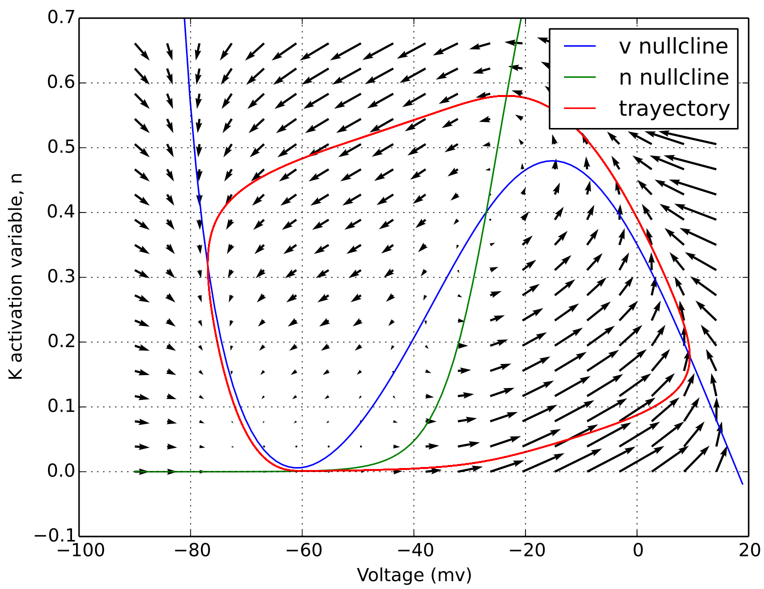


Figure 2: Top: phase space of the INap+IK model with high-threshold parameters. Bottom: voltage trace of the INap+IK model (green trace), and its approximation by a QIF model (blue trace). Input current 4.65 mA.



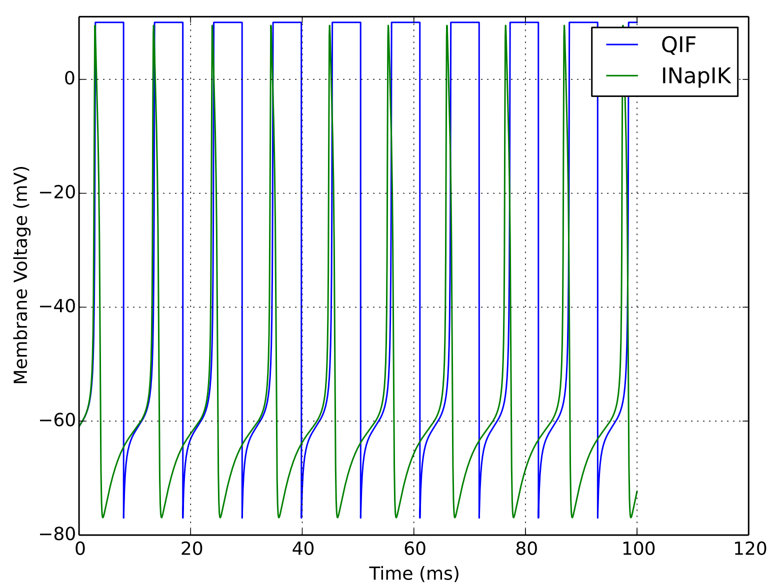
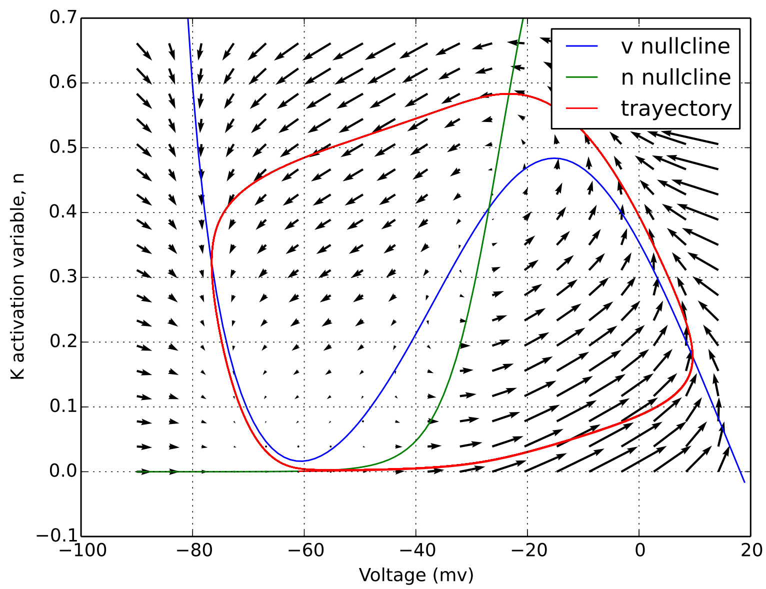


Figure 3: Top: phase space of the INap+IK model with high-threshold parameters. Bottom: voltage trace of the INap+IK model (green trace), and its approximation by a QIF model (blue trace). Input current 6.00 mA.



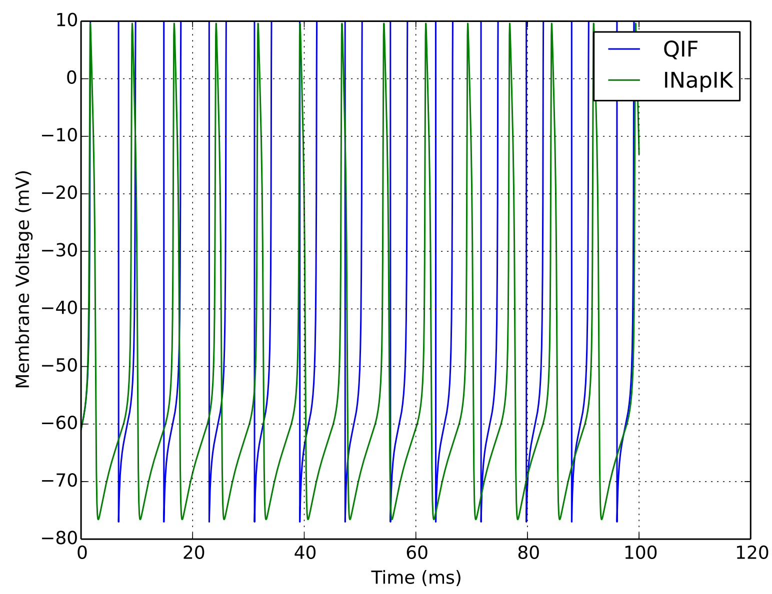
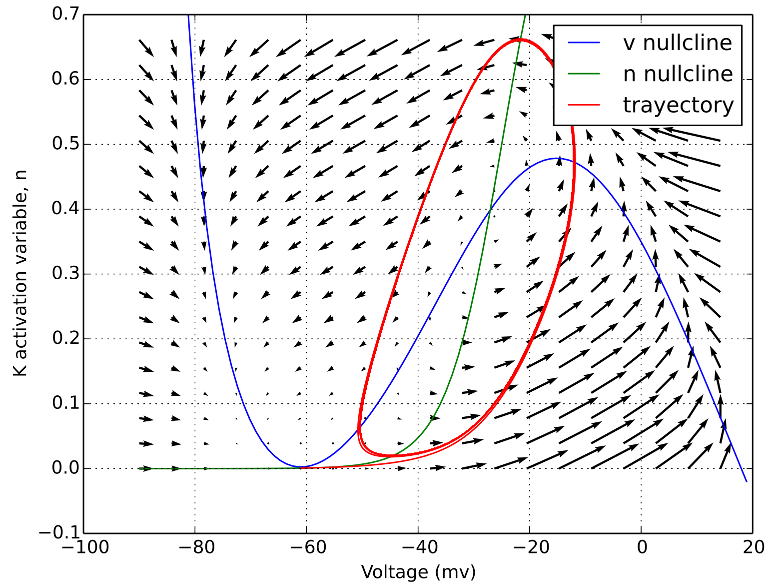


Figure 4: Top: phase space of the INap+IK model with high-threshold parameters. Bottom: voltage trace of the INap+IK model (green trace), and its approximation by a QIF model (blue trace). Input current 9.00 mA.



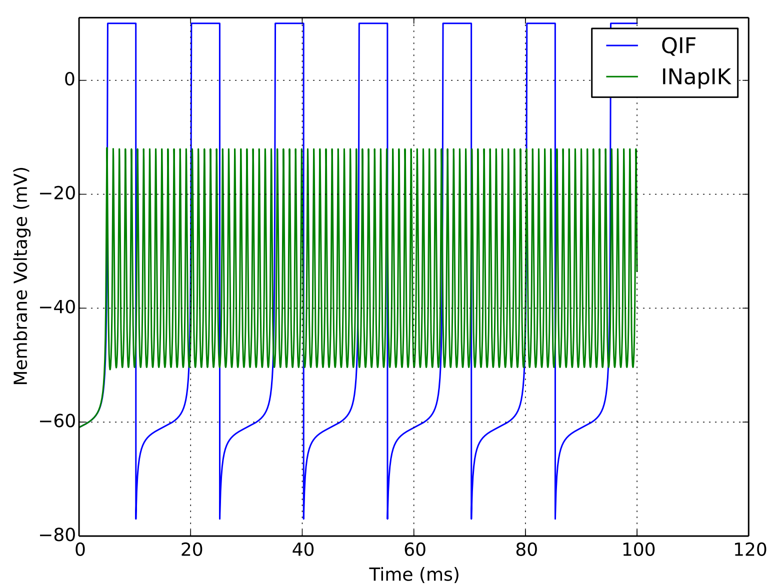


Figure 5: Top: phase space of the INap+IK model with high-threshold parameters and fast K current (i.e.; $\tau(V) = 0.152$).. Bottom: voltage trace of the INap+IK model (green trace), and its approximation by a QIF model (blue trace). Input current 5.00 mA.